Friend- and Enemy-oriented Hedonic Games with Strangers (Extended Appendix B)

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A Appendix B: Section 5 Proofs

Proof for Theorem 5.

Proof. We begin by introducing a set of necessary and sufficient conditions for an N-NS partition to exist for any symmetric FOHGS, briefly observe that the conditions can be checked in polynomial time, then prove that the conditions are necessary and sufficient.

The following conditions are necessary and sufficient for an N-NS partition to exist for any symmetric FOHGS. For all $i \in N$, one of the following must hold:

- $F_i \neq \emptyset$
- $E_i = N \setminus \{i\}$

For any agent $i \in N$, $|F_i \cup E_i \cup S_i| = |N| - 1$, meaning that these conditions can be checked in at worst $O(|N|^2)$ time for any FOHGS instance. We now prove that these conditions are both necessary and sufficient in order for an N-NS partition to exist for any FOHGS instance.

First, we observe that, for any agent $i \in N$ such that $|F_i| > 0$, i will have strictly positive utility in any coalition C where $|C \cap F_i| > 0$, meaning that any partition containing any such coalition is individually rational for i. If $\nexists i \in N : F_i = \emptyset$, then the grand coalition will be individually rational for all $i \in N$ and thus will be N-NS as well since the only way for an individual agent to deviate from the grand coalition is to become a singleton. Now suppose that there exists at least one agent $i \in N : F_i = \emptyset$. Since i has no known friends, the only N-IR coalition for i is to be a singleton since any other coalition provides the possibility of negative utility, so any N-NS partition must have i as a singleton. If S_i is also empty, then there are no problems, as all other agents must be known enemies of i, so they have no possible motivation to join any coalition. On the other hand, if S_i is non-empty, then i will have possible motivation to join any coalition where a stranger exists. Thus, if $F_i = \emptyset$, but $S_i \neq \emptyset$ for any $i \in N$, then an N-NS partition cannot exist. If, instead, for all $i \in N$ either $F_i \neq \emptyset$ or $E_i = N \setminus \{i\}$, then the partition $\gamma = \{\{i \in N : F_i \neq \emptyset\}\}, \bigcup_{j \in N : F_j = \emptyset} \{\{j\}\}$ is N-NS.

Thus, we conclude that N-NS existence can be checked for symmetric FOHGS in polynomial time.

Proof for Theorem 6.

Proof. We construct reductions from Exact Cover by 3 Sets (EC3) similar to those used by Brandt, Bullinger, and Tappe (2022) in their proofs that NS existence is NP-complete for FOHG and EOHG. First, we observe that the reductions used by Brandt, Bullinger, and Tappe (2022) in the proofs showing the hardness of NS existence can't be used as-is. Thus, we modify the reductions used by Brandt, Bullinger, and Tappe (2022) to prove the hardness of N-IS existence in FOHGS and EOHGS.

N-IS existence for asymmetric FOHGS. Given an EC3 instance (R, S), we produce a FOHGS (N, F, E, S) such that (R, S) has an exact cover if and only if (N, F, E, S) has an N-IS partition. For each $r \in R$, let $S_r = \{s \in S : r \in s\}$ and $n_r = |S_r|$. The agent set N will be defined by $N = \{d\} \cup \bigcup_{s \in S} A^s \cup \bigcup_{r \in R}(\{c_1^r, c_2^r\}, \cup \{b_i^r : i \in [n_r - 1]\})$ where $A^s = \{a_r^s : r \in s\}$ for each $s \in S$. Agent relations are defined as follows:

- For each $s \in S$, $a \neq a' \in A^s$: $a' \in F_a$
- For each $r \in R$, $s \in S_r$, $i \in [n_r - 1]$: $b_i^r \in F_{a_r^s}$ $a_r^s, d \in F_{b_i^r}$ $c_1^r \in S_{b_i^r}$ $b_i^r \in S_d$
- For each $r \in R$: $c_2^r \in F_{c_1^r}$ $c_1^r, d \in F_{c_2^r}$ $c_2^r \in S_d$
- For each $r \in R$, $s \in S_r$, $r' \neq r \in s$: $c_1^r \in F_{a_r^s}$ $c_1^{r'} \in S_{a_r^s}$ $a_r^s \in F_{c_1^r}$
- For each $r \in R$, $s \in S_r$, $i \in [n_r - 1]$, $r' \neq r \in s$: $a_s^{r'} \in S_{b_i^r}$
- All relations not otherwise defined are enemy relations

Since the number of agents is polynomial in the size of (R, S), this reduction can be computed in polynomial time. Next, we show that if $\exists S' \subseteq S$ such that S' is an exact cover of R, then an N-IS partition exists for (N, F, E, S).

Suppose there exists some $S' \subseteq S$ that is an exact cover of R, we can construct an N-IS partition γ for (N, F, E, S). For each $s \in S'$ let A^s form a single coalition in γ. Supposing that S' is an exact cover, for each $r \in R$, it must hold that $n_r - 1 = |S_r \setminus S'|$. For each $r \in R$, $i \in [n_r - 1]$, let s_i denote an arbitrary member $s \in S_r \setminus S'$ and let $\{b_i^r, a_{s_i}^r\} \in \gamma$. For each $r \in R$ let $\{c_1^r, c_2^r\} \in \gamma$. Let $\{d\}$ be a singleton in γ.

Since d has no known friends, they lack necessary motivation to join any coalition. For each $r \in R$, $i \in [n_r - 1]$, d views b_i^r and c_2^r as strangers, meaning that d has possible motivation to join any coalition containing one or more b_i^r or c_2^r agents. For each $r \in R$, $i \in [n_r - 1]$, $s \in S_r \setminus S'$, $a_{s_i}^r$ views d as a known enemy, so d cannot get possible permission to join the coalition $\{a_{s_i}^r, b_i^r\}$. For each $r \in R$, c_1^r views d as a known enemy, so d cannot get possible permission to join the coalition $\{c_1^r, c_2^r\}$. Thus, d cannot get possible permission to join any of the coalitions it has possible motivation to join.

For each $r \in R$, c_1^r and c_2^r are in a coalition with one known friend and no enemies. c_2^r is possibly indifferent between their current coalition and leaving to join d so they lack any possible motivation to leave c_1^r to join d. For each $s \in S_r$ c_1^r has at most one known friend and at least one known enemy in each coalition containing $a_r^s \in A^s$, so they lack possible motivation to join any coalition containing at most one a_r^s agent. Thus, c_1^r and c_2^r lack possible motivation to join any other coalition.

For each $r \in R$, $i \in [n_r - 1]$, b_i^r is in a coalition with one known friend and no enemies, so they are possibly indifferent between their current coalition and leaving to join d, so they lack possible motivation to deviate to join d. For each $s \neq s_i \in S_r$, a_s^r is in a coalition with at least one agent who is a known mutual enemy of b_i^r or who views b_i^r as a known enemy, so b_i^r lacks possible permission, possible motivation, or both to join any such coalition. b_i^r views c_1^r as a stranger, but c_1^r views b_i^r as a known enemy, so b_i^r cannot get permission to join any coalition containing c_1^r. Thus b_i^r lacks motivation, permission, or both to join any other coalition.

For each $r \in R$, $s \in S'$, a_s^r is in a coalition both agents $a \neq a_s^r \in A^s$. No two other friends of a_s^r are in the same coalition, so a_s^r lacks possible motivation to move to any other coalition. For each $r \in R$, $i \in [n_r - 1]$, $s_i \in S_r \setminus S'$, $a_{s_i}^r$ is in a coalition with b_i^r and no other agents. Since each $a \neq a_{s_i}^r \in A^{s_i}$ is not contained in any $A^s : s \in S'$, they are paired with some agent b who is a known enemy of $a_{s_i}^r$, so $a_{s_i}^r$ lacks possible motivation to join any coalition containing some $a \neq a_{s_i}^r \in A^s$. c_1^r is paired with c_2^r who is a known mutual enemy of a_s^r, so a_s^r lacks both possible motivation and possible permission to leave their current coalition to join c_1^r. Thus, each agent in this partition lacks possible motivation to deviate, possible permission to deviate to a coalition they want to join, or both, meaning that the partition is N-IS. We conclude that if an exact cover exists for (R, S), then an N-IS partition must exist for (N, F, E, S).

Next, we show that if there exists some N-IS partition γ for (N, F, E, S), then there must exist some $S' \subseteq S$ such that S' is an exact cover of R.

Because d has no known friends, they must be a singleton in any N-IS partition, as otherwise they have possible motivation to deviate to become a singleton. For each $r \in R$, c_2^r must be in a coalition with c_1^r and no enemies, or they will have possible motivation and possible permission to join d. For each $r \in R$, $s \neq s' \in S_r$, a_s^r and $a_{s'}^r$ must be in separate coalitions, or c_1^r will have possible motivation and possible permission to join the coalition, thereby leaving c_2^r as a singleton which could possibly join d. For each $r \in R$, $i \in [n_r - 1]$, b_i^r must be paired with at least one $a_{s_i}^r$ or b_i^r will have possible motivation and possible permission to join d. For each $r \in R$, $i \in [n_r - 1]$, b_i^r will have possible motivation and permission to deviate from any matching which places them in a coalition with one or more agents b_j^r, where $j \neq i \in [n_r - 1]$ without also including at least one additional agent $a_{s_j}^r$. We've established that no N-IS partition can place two agents $a_s^r, a_{s'}^r$ where $s \neq s'$ in the same coalition for each $r \in R$. Thus, for each $r \in R$, $i \in [n_r - 1]$, b_i^r must be in a coalition with exactly one agent $a_{s_i}^r$.

For each A^s group, let a_s^r, $a_s^{r'}$, and $a_s^{r''}$ refer to its three members. Either, each member must be in a coalition with at least one b agent who is a known friend, or $A^s \in \gamma$ must hold. Suppose there existed some group A^s such that one member a_s^r is contained in a coalition $\{a_s^r, b\}$ where a_s^r and b are known friends, while $a_s^{r'}$ and $a_s^{r''}$ are singletons. Both $a_s^{r'}$ and $a_s^{r''}$ have necessary motivation and permission to join each other to form a pair instead of singletons. Thus, no partition containing $\{a_s^r, b\}, \{a_s^{r'}\}$, and $\{a_s^{r''}\}$ for any A^s group can be N-IS. If we suppose instead that $\{a_s^r, b\} \in \gamma$ and $\{a_s^{r'}, a_s^{r''}\} \in \gamma$, then a_s^r has necessary motivation and permission to leave b to join $\{a_s^{r'}, a_s^{r''}\}$, thereby leaving b as a singleton and forming A^s. Thus, no partition containing $\{a_s^r, b\}$ and $\{a_s^{r'}, a_s^{r''}\}$

for any A^s group can be N-IS. We conclude that no N-IS partition exists where, for some group A^s, a^r_s is paired with some known friend b, while $a^{r'}_s$ and $a^{r''}_s$ are not. Suppose instead that two members of A^s, a^r_s and $a^{r'}_s$ were paired with known friends b and b' respectively, leaving $a^{r''}_s$ as a singleton. Since $a^{r''}_s$ is viewed as a stranger by both b and b', $a^{r''}_s$ has necessary motivation and possible permission to join both $\{a^r_s, b\}$ and $\{a^{r'}_s, b'\}$, so the partition cannot be N-IS. Further, neither partition resulting from such deviations, $\gamma' \ni \{a^r_s, b, a^{r''}_s\}$ and $\gamma'' \ni \{a^{r'}_s, b', a^{r''}_s\}$, is N-IS because b (resp. b') has possible motivation and permission to leave to join d, and $a^{r'}_s$ (resp. a^r_s) has possible motivation and permission to leave $\{a^{r'}_s, b'\}$ (resp. $\{a^r_s, b\}$) and join $\{a^r_s, b, a^{r''}_s\}$ (resp. $\{a^{r'}_s, b', a^{r''}_s\}$). We conclude that no N-IS partition exists where, for some group A^s, two agents a^r_s and $a^{r'}_s$ are paired with known friends b and b', while the remaining agent $a^{r''}_s$ is not. Suppose now that all three members of A^s are paired with a known friend b, b', and b'' respectively. Supposing that for each $r \in R$, c^r_1 is paired with c^r_2, each $a \in A^s$ has at most one friend and at least one known enemy in every other coalition in the partition, so they lack possible motivation to join any other coalition. Suppose now that $A^s \in \gamma$. Supposing that for each $r \in R$ and $i \in [n_r - 1]$, b^r_i is paired with some agent $a^r_{s_i}$ such that $s_i \neq s \in S$, each $a \in A^s$ has at most one known friend and at least one known enemy in every other coalition, so they lack possible motivation to join any other coalition. Thus, for each group A^s, either $A^s \in \gamma$ must hold or each agent $a \in A^s$ must be paired with at least one known friend b.

For each $r \in R$, $|\{A^s : a^r_s \in S^s\}| = n_r$ and $|\{b^r_i : i \in [n_r - 1]\}| = n_r - 1$. This means that at most $n_r - 1$ of the n_r agents a^r_s can be paired with a known friend b^r_i, so for each $r \in R$ there must exist some group A^s such that $A^s \in \gamma$ and $a^r_s \in A^s$ if an N-IS partition exists.

Combining the requirements that we've established thus far, we conclude that any N-IS partition γ for (N, F, E, S) based upon some EC3 instance (R, S) must be structured as follows. For each $r \in R$, $\{c^r_1, c^r_2\} \in \gamma$ must hold and there must exist exactly one coalition $A^s \in \gamma$ such that $a^s_r \in A^s$. For each $r \in R$, $i \in [n_r - 1]$, $\{b^r_i, a^r_{s_i}\} \in \gamma$ must hold for some $s_i \in S_r$. For each group A^s, either $A^s \in \gamma$ must hold, or each agent $a \in A^s$ must be paired with exactly one known friend b so that $\{a, b\} \in \gamma$ holds. Agent d must be a singleton so $\{d\} \in \gamma$ holds. These structural requirements indicate that any N-IS partition (N, F, E, S) exhibits the same structure as a partition that results from converting an exact cover $S' \subseteq S$ of (R, S) into an N-IS partition for (N, F, E, S). Thus, we conclude that if an N-IS partition exists for (N, F, E, S), then an exact cover must exist for (R, S).

We conclude that an exact cover exists for (R, S) if and only if an N-IS partition exists for (N, F, E, S). **Therefore, N-IS existence for asymmetric FOHGS is NP-complete.**

N-IS existence for asymmetric EOHGS. Given an EC3 instance (R, S) we produce an EOHGS instance (N, F, E, S) such that (R, S) has an exact cover if and only if (N, F, E, S) has an N-IS partition. For each $r \in R$, let $S_r = \{s \in S : r \in s\}$ and $n_r = |S_r|$ The agent set N will be defined by $N =$

$\{d\} \cup \bigcup_{s \in S} A^s \cup \bigcup_{r \in R}(\{b_i^r : i \in [n_r - 1]\})$ where $A^s = \{a_s, a_{r_1}^s, a_{r_2}^s, a_{r_3}^s\}$ for each $s = \{r_1, r_2, r_3\} \in S$. Agent relations are defined as follows:

- For each $s \in S$, $a \neq a' \in A^s$: $a' \in F_a$
- For each $r \in R$, $s \in S_r$, $i \in [n_r - 1]$: $b_i^r \in F_{a_r^s}$ $a_r^s, d \in F_{b_i^r}$ $b_i^r \in S_d$
- All relations not otherwise defined are enemy relations

This reduction can be computed in polynomial time. We now show that if an exact cover $S' \subseteq S$ of (R, S) exists, then an N-IS partition exists for (N, F, E, S).

Supposing there exists exact cover $S' \subseteq S$ of (R, S), we construct an N-IS partition γ for (N, F, E, S). For each $s \in S'$, let A^s be a coalition in γ. Since S' is an exact cover, it must hold that $|S_r \setminus S'| = n_r - 1$, so for $i \in [n_r - 1]$ we let s_i denote an arbitrarily chosen $s \in S_r \setminus S'$ and let $\{b_i^r, a_{s_i}^\} \in \gamma$. Let $\{d\}$ be a singleton in γ, and for each $s \notin S'$ let $\{a_s\}$ be a singleton in γ.

First, we observe that d has no known friends, so they have no necessary motivation to join any other coalition. The only strangers that d has are the b agents, each of whom is paired with the a agent who is mutual enemies with d. Thus, d lacks possible motivation and possible permission to join any other coalition.

For each $r \in R$, $i \in [n_r - 1]$, b_i^r is in a coalition with exactly one known friend and no other agents. While b_i^r is possibly indifferent between their assigned coalition and leaving to join d, they do not have possible motivation to deviate and join d. For each $s \neq s_i \in S_r$, a_s^r is in a coalition with at least one known mutual enemy of b_i^r, so b_i^r has no possible motivation to join another coalition with a known friend.

For each $r \in R$, $s \in S_r$, a_s^r is either in a coalition with some known mutual friend b_i^r, for some $i \in [n_r - 1]$, or in a coalition with known mutual friends $a_s, a_s^{r'}$, and $a_s^{r''}$. If they are in a coalition with some b_i^r, then they are indifferent between their current assignment and leaving to join a_s, so they lack possible motivation to join a_s; the other friends of a_s^r are each paired with some known mutual enemy of a_s^r, b, so a_s^r lacks possible motivation and possible permission to leave b_i^r to join any of the coalitions with their other friends. If a_s^r is in the coalition A^s, then all other coalitions have at most one known mutual friend and have at least one mutual enemy, so a_s^r has neither possible motivation nor possible permission to join another coalition. Thus, the partition derived from exact cover S' is N-IS.

Next, we show that if an N-IS partition exists for (N, F, E, S), then an exact cover exists for (R, S).

First, we observe that, because d has no known friends, they must be a singleton in any N-IS partition, since otherwise, they have possible motivation to leave and become a singleton.

For each $s' \neq s \in S$, no N-IS partition can place $a \in A^s$ in a coalition with any agent $a' \in A^{s'}$, since a will have necessary motivation to become a singleton. For each $r \in R$, $i \in [n_r - 1$, b_i^r must be in a coalition with at least one agent $a_{s_i}^r \in A^{s_i}$ for some $s_i \in S_r$ or b_i^r will have possible motivation and possible permission to join d. Since no two agents a, a' from $A^s \neq A^{s'}$ respectively can be in the same coalition, b_i^r must be in a coalition with exactly one agent $a_{s_i}^r \in A^{s_i}$ for some $s_i \in S_r$ and no other agents. For each $r \in R$, $s \in S_r$, $a_s^r \in A^s$, we've established that $F_{a_s^r} = \{a \neq a_s^r \in A^s \cup b_i^r \forall i \in [n_r - 1]\}$. Since (N, F, E, S) is an EOHGS, no N-IS partition can place any a_s^r in a coalition with any agent $a \in N \setminus F_{a_s^r}$ since all other agents are known enemies, so any N-IS partition must place a_s^r in a clique with agents from this set. Since for all $r \in R$, $i \neq j \in [n_r - 1]$, b_i^r and b_j^r are known mutual enemies, a_s^r can be in a coalition with no more than one b agent, as two or more b agents in a coalition will give both of them necessary motivation to leave to become a singleton and possible motivation and possible permission to join d.

First, we address the case when a_s^r is not paired with a b agent, showing that they must be in A^s unless they are paired with some agent $b \in F_{a_s^r}$. Suppose this was not the case and some N-IS partition exists where a_s^r is not paired with any agent b_i^r and is not in the coalition A^s. Based upon the known friends of a_s^r, the only other individually rational coalitions they could be a member of include being a singleton or being in some coalition $A^{s^*} \subset A^s$. Suppose a_s^r were a singleton. First, we observe that all agents in the set $N \setminus A^s$ are known enemies of the agent a_s, so if the partition is truly N-IS, then a_s must be a singleton or in some coalition $A^{s^*} \subset A^s$. In either case, a_s^r has necessary motivation and permission to join whichever coalition a_s is a member of since a_s^r is known mutual friends with a_s and all known friends of a_s, so a_s^r cannot be a singleton in any N-IS partition. Suppose instead that a_s^r is in some coalition $A^{s^*} \subset A^s$. In this case, $|A^{s^*}| \in \{2, 3\}$, which will provide all agents $a_s^{r'} \in A^s \setminus A^{s^*}$ necessary motivation and permission to leave their current coalition to join A^{s^*} since they will have strictly more friends and no known enemies, ultimately resulting in A^s forming over the course of one or two necessary deviations.

Now we address the case where a_s^r is paired with some agent $b \in F_{a_s^r}$, showing that if a_s^r is paired with a b agent, then all $a_s^{r'}$ must also be paired with b agents and a_s must be a singleton. Suppose that a_s^r is paired with some agent $b \in F_{a_s^r}$, but no other agent $a' \in A^s$ is paired with a similar b agent. As previously described, such a partition will necessarily result in the other members of A^s forming a coalition together over the course of some number of deviations until a_s^r has more known friends and no known enemies in the resulting A^{s^*} coalition than they do in a pair with b, so they will have necessary motivation and permission to leave their partner and join their other friends. In contrast, if all agents $a \in A^s \setminus \{a_s\}$ are paired with some agent $b \in F_a$, then each agent is indifferent between their current coalition and leaving to join a_s, and a_s lacks possible motivation and possible permission to join any of their friends, so they remain as a singleton.

Combining the requirements that we've outlined, we find that any N-IS partition γ must be structured as follows. Agent d must be a singleton $\{d\} \in \gamma$. For each $r \in R$, $i \in [n_r - 1]$, b_i^r must be paired with some agent $a_{s_i}^r \in A^{s_i}$ for some arbitrarily chosen $s_i \in S_r$ so that $\{b_i^r, a_{s_i}^r\} \in \gamma$ For each $r \in R$, there must exist exactly one $s \in S_r$ such that $A^s \in \gamma$. These structural requirements result in a partition that follows the same structure as the partitions that result from converting some exact cover $S' \subseteq S$ into an N-IS partition of (N, F, E, S). Thus, we conclude that if an N-IS partition exists for (N, F, E, S), then an exact cover must exist for (R, S).

We conclude that an exact cover exists for (R, S) if and only if an N-IS partition exists for (N, F, E, S). **Therefore, N-IS existence for asymmetric EOHGS is NP-complete.**

Proof for Theorem 7.

Proof. We begin by introducing a set of necessary and sufficient conditions for an N-CIS partition to exist for any symmetric FOHGS, briefly observe that the conditions can be checked in polynomial time, then prove that the conditions are necessary and sufficient.

The following conditions are necessary and sufficient for an N-CIS partition to exist for any symmetric FOHGS. For all $i \in N$ one of the following must hold:

- $F_i \neq \emptyset$ OR
- $\forall j \in S_i$, $F_j \neq \emptyset$ AND $\exists k : F_k \neq \emptyset \wedge i \in E_k$

For any agent $i \in N$, $|F_i \cup E_i \cup S_i| = |N| - 1$, meaning that these conditions can be checked in at worst $O(|N|^3)$ time for any FOHGS instance. We now prove that these conditions are both necessary and sufficient in order for an N-NS partition to exist for any FOHGS instance.

First, we observe that, for any agent $i \in N$ such that $|F_i| > 0$, i will have strictly positive utility in any coalition C where $|C \cap F_i| > 0$, meaning that any partition containing any such coalition is individually rational for i. Further, by placing all $i \in N : F_i \neq \emptyset$ in a coalition together, we ensure that none of these agents can leave, since they lack possible permission from their friend(s). Agents who have no known friends have possible motivation and permission to leave any non-singleton coalition, so any N-CIS partition places all agents $i \in N$ where $F_i = \emptyset$ as singletons. If any two agents $i \neq j \in N$ where $F_i = \emptyset = F_j$ are strangers with each other, then they both have possible motivation and possible permission to join the other to form a pair, meaning that the partition is not N-CIS; however, since they have no known friends, they have possible motivation and possible permission to leave any non-singleton coalition to become a singleton, so no N-CIS partition can place them into the same coalition. Thus no N-CIS partition can exist when two agents with empty sets of known friends are strangers with each other. Given any two agents $i \neq j \in N$ where $F_i = \emptyset$, $F_j \neq \emptyset$, and $j \in S_i$, i has possible motivation to join the coalition containing all agents with known friends; however, this is not sufficient to preclude the existence of an N-CIS partition. In order for some agent $i : F_i = \emptyset$ to have possible motivation and possible permission to join the set of agents with known friends, $\{j \in N : F_j \neq \emptyset\} \subseteq S_i$ must hold. Thus, even if i is strangers with some agent(s) with known friends, so long as $\exists j \neq i \in N : F_j \neq \emptyset$ and $i \in E_j$, i cannot get permission to join the coalition of agents with known friends. Thus if each agent $i \in N$ either has $F_i \neq \emptyset$ or $\forall j \in S_i$, $F_j \neq \emptyset$ and $\exists k : F_k \neq \emptyset \wedge i \in E_k$, then the partition $\gamma = \{\{i \in N : F_i \neq \emptyset\}\}, \bigcup_{j \in N : F_j = \emptyset} \{\{j\}\}$ is N-CIS.

The conditions that are necessary and sufficient for N-CIS existence in symmetric FOHGS are also necessary and sufficient for symmetric EOHGS and can similarly be checked in at worst $O(|N|^3)$ time. Although the utility values for friends and enemies are different, equivalent logic proves that the same conditions guarantee the existence of an N-CIS partition for symmetric EOHGS.

Proof for Theorem 8.

Proof. We construct another reduction from EC3 inspired by proofs by Brandt, Bullinger, and Tappe (2022). Given a EC3 instance (R, S), $\forall s \in S$, $r_1, r_2, r_3 \in s$ let $A^s = \{a_s, a_s^{r_1}, a_s^{r_2}, a_s^{r_3}\}$. $\forall r \in R$, $\forall i \in [n_r - 1]$, we create b_i^r. $\forall r \in R$, we create c_1^r, c_2^r, c_3^r. We create d, e_1, e_2, f, g_1, and g_2. $\forall s \in S$, we create h_s^1, h_s^2.

Next we define relationships.

- $\forall s \in S$:
  - $F_{h_s^1} = \{h_s^2\}$
  - $S_{h_s^1} = \{a_s \cup \bigcup_{\forall s' \neq s}\{a_s^r : r \in s'\} \cup \{f\}$
  - $S_{h_s^2} = \{d\}$
- $F_{g_1} = \{g_2\}$
- $S_{g_1} = \bigcup_{s \in S, r \in s}\{a_s^r\} \cup \{f\}$
- $S_f = \{e_1, g_1\} \cup \bigcup_{s \in S}\{h_s^1\}$
- $S_{e_2} = S_{g_2} = \{d\}$
- $F_{e_1} = \{e_2\}$
- $S_{e_1} = \bigcup_{s \in S}\{a_s\} \cup \bigcup_{r \in R}\{c_1^r\} \cup \{f\}$
- $S_d = \bigcup_{\forall r \in R, \forall i \in n_r}\{b_i^r\} \cup \bigcup_{\forall r \in R}\{c_2^r, c_3^r\} \cup \{e_2, g_2\} \cup \bigcup_{s \in S}\{h_s^2\}$
- $\forall r \in R$:
  - $F_{c_1^r} = \{c_2^r, c_3^r\}$
  - $S_{c_1^r} = \bigcup_{\forall s \in S_r}\{a_s^r\} \cup \bigcup_{\forall s \in S}\{a_s\} \cup \bigcup_{\forall i \in n_r}\{b_i^r\} \cup \{e_1\} \cup \{g_1\}$
  - $S_{c_2^r} = S_{c_3^r} = \{d\} \cup \{e_1\} \cup \{g_1\}$
- $\forall r \in R$, $\forall i \in [n_r - 1]$:
  - $S_{b_i^r} = \{d\} \cup \bigcup_{\forall r \in R}\{c_1^r\} \cup \{e_1\} \cup \{g_1\} \cup \bigcup_{s \in S}\{h_s^1\}$
- $\forall s \in S$:
  - $F_{a_s} = \bigcup_{r \in s}\{a_s^r\}$
  - $S_{a_s} = \bigcup_{r \in R}\{c_1^r\} \cup \{e_1, h_s^1\}$
- $\forall s \in S$, $\forall r \in s$:
  - $F_{a_s^r} = \bigcup_{a \neq a_s^r \in A^s}\{a\} \cup \bigcup_{\forall i \in [n_r - 1]}\{b_i^r\}$
  - $S_{a_s^r} = \bigcup_{r' \in s}\{c_1^{r'}\} \cup \{e_1\} \cup \{g_1\} \cup \bigcup_{s \in S}\{h_s^1\}$

All relationships not otherwise defined are enmity.

Given an exact cover $S' \subseteq S$, we construct an N-CIS partition γ. $\forall s \in S'$ let A^s be a coalition in γ. Since S' is an exact cover, this leaves $n_r - 1$ sets $s \in S_r \setminus S'$ for each $r \in R$. Next, $\forall i \in [n_r - 1]$ let s_i denote and arbitrary $s \in S_r \setminus S'$ and let $\{a_{s_i}^r, b_i^r\}$ be a coalition in γ. For each $s \notin S'$, let $\{a_s\}$ be a coalition in γ. $\forall r \in R$, let $\{c_1^r, c_2^r, c_3^r\}$ be a coalition in γ. $\forall s \in S$, let $\{h_s^1, h_s^2\}$ be a coalition in γ. Let $\{e_1, e_2\}$, $\{g_1, g_2\}$, $\{d\}$, and $\{f\}$ all be coalitions in γ.

Now let's evaluate γ to see whether it's N-CIS. $\forall s \in S$, h_s^1 has possible motivation to join, $\forall s^* \in S' : s^* \neq s$, A^{s^*}, but lacks possible permission to join since a_{s^*} views h_s^1 as a known enemy. $\forall s \in S$, h_s^2 has necessary motivation to leave and possible motivation and permission to join d, but they lack possible permission to leave, because they're seen as a friend by h_s^1. g_2 has necessary motivation to leave and possible motivation and permission to join d, but they lack possible permission to leave, because they're seen as a friend by g_1. g_1 has possible motivation to join, $\forall s \in S'$, A^s, but lacks possible permission to join due to the presence of known mutual enemies. f has possible motivation to join $\{e_1, e_2\}$, $\{g_1, g_2\}$, and, $\forall s \in S$,$\{h_s^1, h_s^2\}$, but lacks possible permission to join any of these coalitions since each contains a known mutual enemy of f. e_2 has necessary motivation to leave and possible motivation and permission to join d, but they lack possible permission to leave, because they're seen as a friend by e_1. e_1 lacks possible motivation to join any other coalition, because no other coalition contains more than one friend or stranger of e_1. d has possible motivation to join $\{g_1, g_2\}$ or $\{e_1, e_2\}$, but lacks possible permission, because they're mutual enemies with g_1 and e_1 respectively. $\forall r \in R$, d has possible motivation to join the coalition $\{c_1^r, c_2^r, c_3^r\}$ since d is mutual strangers with c_2^r and c_3^r, but lacks possible permission to join since they're mutual known enemies with c_1^r. $\forall r \in R$, $i \in [n_r - 1]$, d has possible motivation to join the $a - b$ coalition containing b_i^r since they're mutual strangers with b_i^r, but lacks possible permission to join since they're mutual known enemies with all agents $a \in A$. $\forall r \in R$, c_2^r and c_3^r have necessary motivation to leave $\{c_1^r, c_2^r, c_3^r\}$ to either be a singleton or possibly to join d, $\{e_1, e_2\}$, or $\{g_1, g_2\}$, but lack possible permission to leave since c_1^r views them as a known friend. $\forall r \in R$, $i \in [n_r - 1]$, c_1^r lacks possible motivation to join any other coalition in γ since they have two known friends in $\{c_1^r, c_2^r, c_3^r\}$ and at most two strangers in any other coalition in γ. $\forall r \in R$, $i \in [n_r - 1]$, $s : \{a_s^r, b_i^r\} \in \gamma$, b_i^r has necessary motivation to leave $\{a_s^r, b_i^r\}$ and become a singleton or possibly join d, $\{c_1^r, c_2^r, c_3^r\}$, $\{e_1, e_2\}$, $\{g_1, g_2\}$, or, $\forall s \in S$, $\{h_s^1, h_s^2\}$, but lacks possible permission to leave since a_s^r views b_i^r as a known friend. $\forall s \notin S'$, $r \in s$, a_s^r lacks possible motivation to join any other coalition since they're in a coalition with one known friend and no enemies, while all other coalitions they could possibly join contain at most one stranger or known friend and, except for $\{a_s\}$, contain at least one known enemy. $\forall s \notin S'$, a_s has possible motivation to join $\{h_s^1, h_s^2\}$ and, $\forall r \in s$ has necessary motivation to join $\{a_s^r, b_i^r\}$ where $i \in [n_r - 1]$ and possible motivation to join $\{c_1^r, c_2^r, c_3^r\}$, but lacks possible permission to join any of these coalitions since each of them contains at least one agent that views a_s as a known enemy. $\forall s \in S'$, all agents $a \in A^s$ lack possible motivation and permission to move to any other coalition, since they have three known mutual friends in A^s and all other coalitions have at most one known friend or stranger. We conclude that, for all agents $i \in N$, i lacks possible motivation to deviate, permission to leave their assigned coalition, permission to enter a desirable destination coalition, or some combination of all three conditions. Thus, γ is N-CIS for (N, F, E, S).

Next we prove that if an N-CIS partition exists, then an exact cover exists. Since f and d have no known friends, both agents have possible motivation and permission to leave any non-singleton coalition they're a part of. Thus $\{f\} \in \gamma$ and $\{d\} \in \gamma$ must hold if γ is N-CIS.

$\forall s \in S$, h_s^2 has no known friends, so they have possible motivation to leave any non-singleton coalition, but if they're left as a singleton, they have possible motivation and permission to join d. Since the only agent who views h_s^2 as a known friend is h_s^1, h_s^2 and h_s^1 must be in the same coalition so h_s^2 lacks possible permission to leave. Since all agents $a \in N \setminus \{h_s^1, h_s^2\}$ are viewed as a known enemy by h_s^1, h_s^2, or both, no agent has possible permission to join any coalition containing h_s^1 and h_s^2. Since h_s^2 is the only known friend of h_s^1, if any agent besides f is in the same coalition as h_s^1 and h_s^2, then h_s^1 has possible motivation and permission to leave and join f. Since f must be a singleton, $\{h_s^1, h_s^2, f\} \in \gamma$ cannot hold if γ is N-CIS. Since h_s^1 and h_s^2 must be in the same coalition and no other agent can be in the same coalition if γ is N-CIS, $\{h_s^1, h_s^2\} \in \gamma$ must hold if γ is N-CIS. Analogous logic suffices to prove that $\{e_1, e_2\} \in \gamma$ and $\{g_1, g_2\} \in \gamma$ must hold if γ is N-CIS.

$\forall r \in R, i \in [n_r - 1]$, b_i^r has no known friends, so they have possible motivation to leave any non-singleton coalition, but if they're a singleton, they have possible motivation and permission to join d. Since the set of agents $\{a_s^r : s \in S_r\}$ are the only agents who view b_i^r as a known friend, b_i^r must be in a coalition with at least one a_s^r, so b_i^r lacks possible permission to leave. We later show that b_i^r must be in a coalition with exactly one a_s^r agent and no others, but first must establish other necessary structural conditions.

$\forall r \in$, the agents c_2^r and c_3^r have no known friends, so both have possible motivation to leave any non-singleton coalition, but if either one or both of them are singletons, they have possible motivation and permission to join d, so $\{c_2^r\} \in \gamma$ and $\{c_3^r\} \in \gamma$ cannot hold if γ is N-CIS. Since c_1^r is the only agent who views c^r2 and c_3^r as known friends, c^r2 and c_3^r must be in a coalition with c_1^r if γ is N-CIS. Further, since the only known friends of c_1^r are c_2^r and c_3^r, c_1^r has possible motivation and permission to leave any non-singleton coalition that doesn't contain c_2^r or c_3^r.

Let $a_s^{r_1}$ and $a_{s'}^{r_2}$ be two agents selected arbitrarily from the set $\bigcup_{s \in S, r \in s} \{a_s^r\}$ such that $a_s^{r_1} \neq a_{s'}^{r_2}$ holds. Let C be a coalition such that $a_s^{r_1} \in C$ and $a_{s'}^{r_2} \in C$. Given that $\{d\}, \{e_1, e_2\}, \{f\}, \{g_1, g_2\}$, and $\forall s \in S$ $\{h_s^1, h_s^2\}$, must be coalitions in γ if γ is N-CIS, let each of these coalitions be members of γ. Since g_1 is in a coalition with only one known friend, they have possible motivation to join C since they're mutual strangers with $a_s^{r_1}$ and $a_{s'}^{r_2}$. Since g_1 has necessary permission to leave $\{g_1, g_2\}$, γ cannot be N-CIS if g_1 has possible permission to join C. If γ is N-CIS, the set agents who view g_1 as a known enemy and can also be a member of C is $\bigcup_{s \in S} \{a_s\}$. Thus, if γ is N-CIS, then C must contain some agent $a_s^* \in \bigcup_{s \in S} \{a_s\}$ so g_1 lacks possible permission to join C.

For some $s \in S$, $r \in R$, let C be a coalition containing a_s and c_1^r. Given that $\{d\}, \{e_1, e_2\}, \{f\}, \{g_1, g_2\}$, and $\forall s \in S$ $\{h_s^1, h_s^2\}$, must be coalitions in γ if γ is N-CIS, let each of these coalitions be members of γ. Since e_1 only has one known friend in their coalition, they have possible motivation to join C since they're mutual strangers with a_s and c_1^r. Since e_1 has necessary permission to leave $\{e_1, e_2\}$, γ cannot be N-CIS if e_1 has possible permission to join C. Since the set of agents who view e_1 as a known enemy is $\{d, e_2, g_1, g_2\} \cup \bigcup_{s \in S}\{a_s\}$, no agent that views e_1 as a known enemy can be a member of C if γ is N-CIS since each agent is required to be in another coalition. Thus, if C contains a_s and c_1^r and γ is N-CIS, then $C \in \gamma$ cannot hold. By the same logic, $\forall s \neq s' \in S$, a_s and $a_{s'}$ cannot be in the same coalition if γ is N-CIS.

For some $s \in S$, $r \in s$, $i \in [n_r - 1]$, let C be a coalition containing a_s and b_i^r. We show that $C \in \gamma$ cannot hold if γ is N-CIS. Suppose there did exist some N-CIS partition γ such that $C \in \gamma$. Given that $\{d\}, \{e_1, e_2\}, \{f\}, \{g_1, g_2\}$, and $\forall s \in S$ $\{h_s^1, h_s^2\}$, must be coalitions in γ if γ is N-CIS, let each of these coalitions be members of γ. Given that $\forall r \in R$, c_1^r, c_2^r, and c_3^r must be in the same coalition if γ is N-CIS, let c_1^r, c_2^r, and c_3^r be in some coalition $C_r \in \gamma$. If $C = \{a_s, b_i^r\}$, then a_s and b_i^r have necessary motivation and permission to leave, so C must also contain one or more agents that remove the possible motivation, permission, or both for a_s and b_i^r to leave. If $a_s^{r'} \in C : r' \neq r$, then b_i^r has necessary motivation and permission to leave and h_s^1 has possible motivation and permission to join. If $a_{s'}^r \in C : s' \neq s$, then a_s has necessary motivation and permission to leave and c_1^r has possible motivation and permission to join. If $a_s^r \in C$, then a_s and b_i^r lack possible permission to leave, but c_1^r has possible motivation and permission to join. If $a_s^{r'} \in C : r' \neq r$ and $a_{s'}^r \in C : s' \neq s$, then a_s and b_i^r lack possible permission to leave, but c_1^r and h_s^1 have possible motivation and permission to join. Since including agents to deny a_s and b_i^r possible permission to leave results in c_1^r, h_s^1, or both having possible motivation and permission to join, C must also include one or more agents capable of denying c_1^r, h_s^1, or both possible permission to join. The set of agents who view c_1^r as a known enemy and could be members of C is $\bigcup_{s' \in S \setminus S_r, r' \in s'}\{a_{s'}^{r'}\}$, but the inclusion of $a_{s'}^{r'}$ would give h_s^1 possible motivation and permission to join. The set of agents who view h_s^1 as a known enemy and could be members of C is $\bigcup_{s' \neq s \in S}\{a_{s'}\} \cup \bigcup_{r' \in R}\{c_1^{r'}, c_2^{r'}, c_3^{r'}\}$, but we have already shown that neither $\{a_s, a_s'\} \subset C$ nor $\{a_s, c_1^{r'}\} \subset C$ can hold. Thus, $C \in \gamma$ cannot hold if γ is N-CIS.

For some $s \neq s' \in S$, $r \in s'$, let coalition C contain a_s and $a_{s'}^r$. We show that $C \in \gamma$ cannot hold if γ is N-CIS. Suppose there did exist some N-CIS partition γ such that $C \in \gamma$. If $C = \{a_s, a_{s'}^r\}$, then a_s and $a_{s'}^r$ have necessary motivation and permission to leave and h_s^1 has possible motivation and permission to join C, so C must contain one or more agents so a_s and $a_{s'}^r$ lack possible motivation, permission, or both to leave and so h_s^1 lacks possible permission to join. We already know that $a_{s'} \in C$ cannot hold if $C \in \gamma$ and γ is N-CIS. We also know $\forall i \in [n_r - 1]$ $b_i^r \in C$ cannot hold if $C \in \gamma$ and γ is N-CIS. Thus C must include some agent $a_s^{r'} : r' \in s$ and some agent $a_{s'}^{r''} : r'' \neq r \in s$ so a_s and $a_{s'}^r$ lack possible permission to leave. The set of agents who view h_s^1 as a known enemy and could possibly be members of C is $\bigcup_{s' \neq s \in S}\{a_{s'}\} \cup \bigcup_{r' \in R}\{c_1^{r'}, c_2^{r'}, c_3^{r'}\}$, but we've already established that none of these agents can be members of C if $C \in \gamma$ and γ is N-CIS. Thus, $C \in \gamma$ cannot hold if γ is N-CIS.

$\forall r \in R$, $s \in S_r$ a_s^r has possible motivation and permission to leave any non-singleton coalition which does not contain at least one known friend of a_s^r. $\forall s \in S$

a_s has possible motivation and permission to leave any non-singleton coalition which does not contain at least one known friend $a_s^r \in A^s$. Since, $\forall r \in R$, $\forall s \in S_r$, a_s and a_s^r are mutual friends, both agents can't be singletons in any N-CIS partition γ.

We've shown that $\forall r \in R$, $\forall i \in [n_r - 1]$, b_i^r must be in a coalition with at least one agent $a_s^r : s \in S_r$. We also know that for any $s \neq s' \in S_r$ no two agents a_s^r and $a_{s'}^r$ can be in the same coalition unless either a_s or $a_{s'}$ is present. Thus, each b_i^r must be in a coalition with exactly one agent a_s^r in any N-CIS partition γ.

We now prove that $\forall r \in R$, there must exist at least one $s \in S_r$ such that a_s and a_s^r are in the same coalition in any N-CIS partition. We know that a_s must either be a singleton or in a coalition with at least one known friend, that a_s cannot be in an agent with any agent b_i^r, and that a_s and a_s^r cannot both be singletons. We've established that $\forall r \in R$, $i \in [n_r - 1]$, b_i^r must be in a coalition with exactly one agent a_s^r for some $s \in S_r$. Since $|\{b_i^r : i \in [n_r - 1]\}| + 1 = |\{a_s^r : s \in S_r\}|$, even if every b_i^r is matched with only one a_s^r, there will be one a_s^r agent who remains unmatched. This means that a_s^r must either be a singleton or be in a coalition with another member of A^S. If $\forall r' \neq r \in s$, $a_s^r r'$ is matched with some $b_i^{r'}$, then a_s^r cannot join them, but this also means that a_s cannot be in a coalition with $a_s^{r'}$. Assuming that a_s is not initially matched with a_s^r, both agents have necessary motivation and permission to form a pair. Assuming that a_s and a_s^r do start as a pair, neither agent has possible permission to leave. Thus, in any N-CIS partition γ, $\forall r \in R$, there must exist some $s \in S_r$ such that a_s and a_s^r are in the same coalition. Further, it follows that $\forall r \in R$, $s \in S_r$, if a_s^r is not matched with some b_i^r for some $i \in [n_r - 1]$, then a_s^r and a_s must be in the same coalition.

We now prove that $\forall r \in R$, $s \in S_r$, $i \neq j \in [n_r - 1]$, a_s^r cannot be in a coalition with both b_i^r and b_j^r. Placing a_s^r in a coalition with b_i^r and b_j^r provides c_1^r with possible motivation to join the coalition since c_1^r only has two known friends. Since none of these three agents view c_1^r as a known enemy, the coalition must also contain some agent who views c_1^r as a known enemy to ensure that c_1^r lacks possible permission to join. Given that $\{d\}, \{e_1, e_2\}, \{f\}, \{g_1, g_2\}$, and $\forall s \in S$ $\{h_s^1, h_s^2\}$, must be coalitions in γ if γ is N-CIS, we can't assign any of these agents to the same coalition as a_s^r, b_i^r and b_j^r. Each $a \in A^s \setminus a_s^r$ views c_r^1 as a stranger, as do each b_k^r $\forall k \in [n_r - 1]$, so these agents do not deny c_1^r possible permission to join. Suppose we added $c_1^{r'}, c_2^{r'}, c_3^{r'}$ for some $r' \neq r \in R$. We've previously established that there must be some coalition $s' \in S_{r'}$ such that $a_{s'}^{r'}$ and $a_{s'}$ are in the same coalition. Based on what's already been proven, we can deduce that the coalition containing $a_{s'}^{r'}$ and $a_{s'}$ cannot contain any agent from the set $\{d, f, e_1, e_2, g_1, g_2\} \cup \bigcup s \in S\{h_s^1, h_s^2\}$. We also know that the coalition containing $a_{s'}^{r'}$ and $a_{s'}$ cannot contain any agent $a \in A^{s''} : s'' \neq s' \in S$ or any agent $b \in \bigcup r \in R, i \in [n_r - 1]\{b_i^r\}$. We also know that $a_{s'}$ cannot be in the same coalition as any $c_1^{r''} : r'' \in s'$. This means that the only agents who can be in the same coalition as $a_{s'}^{r'}$ and $a_{s'}$ are other members of $A^{s'}$. Thus, $c_1^{r'}$ has possible motivation and permission to leave $\{a_s^r, b_i^r, b_j^r, c_1^{r'}, c_2^{r'}, c_3^{r'}\}$ and join the coalition containing $a_{s'}^{r'}$ and $a_{s'}$ since it contains two strangers of $c_1^{r'}$ and at most two known enemies. Altogether, there exist no agents who will deny c_1^r possible permission to join and can also be added to the coalition $\{a_s^r, b_i^r, b_j^r\}$ if the resulting partition must be N-CIS. Suppose instead we start with the coalition $\{a_s^r, b_i^r, b_j^r, c_1^r, c_2^r, c_3^r\}$. In this case, as previously established, there must exist some $s' \in S_r$ such that $a_{s'}^r$ and $a_{s'}$ are in the same coalition. This provides c_1^r possible motivation and permission to join the coalition containing $a_{s'}^r$ and $a_{s'}$. Thus, $\forall r \in R$, $s \in S_r$, $i \neq j \in [n_r - 1]$, a_s^r cannot be in a coalition with both b_i^r and b_j^r.

$\forall s \in S$, if a_s is a singleton, then $\forall r \in s$, a_s^r must be in a coalition with exactly one agent b_i^r for some $i \in [n_r - 1]$. Since a_s is a known mutual friend of a_s^r, a_s has necessary motivation to join a_s^r, so a_s^r must be in a coalition with at least one known friend and at least one agent who will deny a_s possible permission to join. Based on the previous findings, we can narrow down the pool of agents that a_s^r can be grouped with to other members of A^s or b_i^r. Since the other A^s members are also mutual friends with a_s, it follows that a_s^r must be paired with b_i^r in order to deny a_s possible permission to join. Since a_s^r can't be in the same coalition as any $a_s^{r'} : r' \neq r \in s$ unless a_s (or some other agent $a_{s'} : s' \in S$), a_s^r can't be with b_i^r and additional friends from A^s. Thus, if a_s if a_s is a singleton, then $\forall r \in s$, a_s^r must be in a coalition with exactly one agent b_i^r for some $i \in [n_r - 1]$.

We now prove that $\forall s \in S$, $r \in s$, if a_s and a_s^r are in the same coalition, then A^s must be in γ. We've shown that if a_s is alone, then $\forall r \in s$ a_s^r must be paired with exactly one b_i^r where $i \in [n_r - 1]$. We've also shown that $\forall r \in R$, there must exist some $s \in S_r$ such that a_s and a_s^r are in the same coalition. We've also shown that $\forall s \in S$, $r \in s$, a_s^r must be in a coalition with a known friend, meaning either at least one agent from A^s or b_i^r. We've also shown that $\forall s \in S$, $r \neq r' \in s$, a_s^r and $a_s^{r'}$ can only be together if a_s is also present. It follows then that if a_s^r is not paired with b_i^r, then they must be in a coalition with a_s. Now suppose there exists some partition γ and some $s \in S$ such that $r_1, r_2, r_3 \in s$ where $\{a_s, a_s^{r_1}\}, \{a_s^{r_2}, b_i^{r_2}\}, \{a_s^{r_3}, b_j^{r_3}\} \in \gamma$. In any such scenario, $a_s^{r_2}$ and $a_s^{r_3}$ have necessary motivation and permission to leave their current coalition and join $\{a_s, a_s^{r_1}\}$. This results in γ' where either $\{a_s, a_s^{r_1}, a_s^{r_2}\}, \{b_i^{r_2}\}, \{a_s^{r_3}, b_j^{r_3}\} \in \gamma'$ or $\{a_s, a_s^{r_1}, a_s^{r_3}\}, \{a_s^{r_2}, b_i^{r_2}\}, \{b_j^{r_3}\} \in \gamma'$. This leaves $a_s^{r_3}$ (resp. $a_s^{r_2}$) with necessary motivation and permission to join $\{a_s, a_s^{r_1}, a_s^{r_2}\}$ (resp. $\{a_s, a_s^{r_1}, a_s^{r_3}\}$), resulting in γ'' where $A^s \in \gamma''$. This, in turn leaves $b_i^{r_2}$ and $b_j^{r_3}$ with possible motivation and permission to join d meaning that γ, γ', and γ'' all are not N-CIS. We conclude that the only way an N-CIS partition can exist if a_s and a_s^r are in the same coalition is if their initial assignment is to A^s. Thus, $\forall s \in S$, $r \in s$, if a_s and a_s^r are in the same coalition, then A^s must be in γ. Combining the logic from the last several paragraphs, we conclude that $\forall s \in S$, $r \in s$, the coalition containing a_s^r must be A^s or $\{a_s^r, b_i^r\}$ for some $i \in [n_r - 1]$ if γ is N-CIS. Since we've just shown that $\forall s \in S$, $r \in s$, if a_s and a_s^r are in the same coalition, then A^s must be in γ, it follows that $\forall r \in R$, there must exist some $s \in S_r$ such that $A^s \in \gamma$. Further, $|S_r| = n_r$, so there can be no more than one $s \in S_r$ such that $A^s \in \gamma$ if γ is N-CIS.

We now prove that $\forall r \in R$, $\{c_1^r, c_2^r, c_3^r\}$ must be a coalition in any N-CIS partition γ. Suppose there exists some $C \in \gamma$ where γ is N-CIS where $\{c_1^r, c_2^r, c_3^r\} \subset C$. Since $\{c_1^r, c_2^r, c_3^r\}$ is a strict subset of C, there must exist a non-empty set of agents $C \setminus \{c_1^r, c_2^r, c_3^r\}$. We now consider which agents could possibly belong to $C \setminus \{c_1^r, c_2^r, c_3^r\}$ if it's a given that γ is N-CIS. Given that $\{d\}, \{e_1, e_2\}, \{f\}, \{g_1, g_2\}$, and $\forall s \in S$ $\{h_s^1, h_s^2\}$, must be coalitions in γ if γ is N-CIS, let each of these coalitions be members of γ. Given that $\forall r \in R$, $i \in [n_r - 1]$, b_i^r must be in a coalition with exactly one agent $a_s^r : s \in S_r$, $\{a_s^r, b_i^r\} \in \gamma$ must hold if γ is N-CIS, let each such coalition be members of γ. Given that $\forall s \in S$, $r \in s$, a_s^r must either be in A^s or $\{a_s^r, b_i^r\}$ for some $i \in [n_r - 1]$ if γ is N-CIS, let each a_s^r belong to one of two such coalitions in γ. After accounting for all of these requirements, the only remaining agents who may be able to belong to C are those belonging to the set $\bigcup_{r' \neq r \in R} \{c_1^{r'}, c_2^{r'}, c_3^{r'}\}$; note that if any such agent belongs to C, then the other two associated agents must also belong to C. We've established that $\forall r \in R$, there exists some $s \in S_r$ such that $A^s \in \gamma$ holds if γ is N-CIS. If there exists some $C \in \gamma$ where $\{c_1^r, c_2^r, c_3^r\} \subset C$ and γ is N-CIS, then there must be at least one r' such that $\{c_1^{r'}, c_2^{r'}, c_3^{r'}\} \subset C$ since these agents are the only agents who are not required to belong to some other coalition in any N-CIS partition. The best-case C that satisfies these requirements is defined by $r \neq r' \in R$ such that $C = \{c_1^r, c_2^r, c_3^r, c_1^{r'}, c_2^{r'}, c_3^{r'}\}$ since it minimizes the number of enemies c_1^r and $c_1^{r'}$ have in C, so let this C be a member of γ. If we compare C to A^s, we see that C has 2 known friends and 3 known enemies for c_1^r while A^s has two mutual strangers and two enemies that c_1^r views as enemies, but both view c_1^r as a stranger, meaning that c_1^r has possible motivation and permission to leave C and join A^s instead. Thus we have a contradiction to the claim that γ is N-CIS. We observe that adding an additional trio $c_1^{r''}, c_2^{r''}, c_3^{r''}$ to C will only reduce the utility c_1^r further, so no such additions will remove the possible motivation and permission for c_1^r to deviate. We also note that $\forall r \in R$ such that $n_r > 1$, for any $s \in S_r$, $i \in [n_r - 1]$ such that $\{a_s^r, b_i^r\} \in \gamma$ holds, c_1^r has possible motivation and permission to join $\{a_s^r, b_i^r\}$ if $\{c_1^r, c_2^r, c_3^r\} \in \gamma$ doesn't hold. Thus we conclude that $\forall r \in R$, $\{c_1^r, c_2^r, c_3^r\}$ must be a coalition in any N-CIS partition γ.

Lastly we show that $\forall s \in S$, h_s^1 lacks possible motivation, permission, or both to join A^s or, for any $s' \neq s \in S$, $A^{s'}$. We've established that h_s^1 must be in a pair with h_s^2 in any N-CIS partition γ. h_s^1 has possible motivation to join any coalition $A^{s'}$ where $s' \neq s$, but lacks possible permission to join since $a_{s'}$ views h_s^1 as a known enemy. h_s^1 lacks possible motivation to join A^s since they have one stranger and three known enemies, even though they have possible permission to join. Further, no agent who views h_s^1 as a stranger has permission to join $\{h_s^1, h_s^2\}$ since h_s^2 views all such agents as known enemies.

Below we summarize the requirements for a FOHGS instance to have an N-CIS partition when it's derived from (R, S) as described at the beginning of the proof:

- $\{d\} \in \gamma$, $\{f\} \in \gamma$, $\{e_1, e_2\} \in \gamma$, $\{g_1, g_2\} \in \gamma$ all hold
- $\forall r \in R$:

  - $\forall i \in [n_r - 1]$:
    * $\exists s \in S_r$ such that $\{a^r_s, b^r_i\} \in \gamma$ holds
  - $\{c^r_1, c^r_2, c^r_3\} \in \gamma$ holds
  - There exists exactly one $s \in S_r$ such that $A^s \in \gamma$ holds
- $\forall s \in S$:
  - $\{h^1_s, h^2_s\} \in \gamma$
  - One of the following holds:
    * $A^s \in \gamma$
    * $\{a_s\} \in \gamma$ and $\forall r \in s$, $\exists i \in [n_r - 1]$ such that $\{a^r_s, b^r_i\} \in \gamma$

Each A^s group corresponds to some $s \in S$, so we construct a candidate cover r^* such that $\forall s \in S : A^s \in \gamma$ let $s \in r^*$ hold. Since, in order for γ to be N-CIS, $\forall r \in R$, there must exist exactly one $s \in S_r$ such that $A^s \in \gamma$, the set $r^* \subseteq S$ such that $\forall s \in S : A^s \in \gamma$ corresponds to an exact cover of (R, S). This indicates that the FOHGS instance has an N-CIS partition if and only if an exact cover exists for the EC3 instance it's based on. **Thus, N-CIS existence for asymmetric FOHGS is NP-complete.**

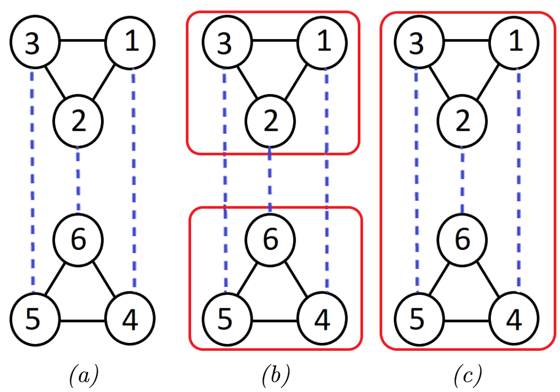

Fig. 1. FOHGS with no N-CS partition.

Proof for 9.

Proof. Consider a FOHGS based on the graph shown in Figure 1 (a).

First, we rule out several types of partitions that are all possibly or necessarily blocked, and thus cannot be N-CS.

Any partition which does not place all members of a known friend trio in the same coalition is possibly blocked by the 3-clique coalition of the friend trio. To prove this, we first observe that regardless of how stranger relations are resolved, a coalition composed solely of a 3-clique of known friends provides 12 utility. Now, suppose that all strangers become enemies. In such a scenario, the only friends an agent has are the members of their 3-clique of known friends. If all three clique members are singletons, then they can increase their utility from 0 to 6 by forming a pair with another clique member and from 0 to 12 by forming a coalition with all three clique members. If two clique members are in a pair and the third is a singleton, all three can increase their utility from either 6 to 12 or 0 to 12 by forming a coalition with all three clique members.

Next, suppose that at least one stranger pair becomes friends. First, we observe that each agent has exactly one stranger, so each agent can gain at most one friend through strangers becoming friends. Suppose that all three members of a 3-clique of known friends are in separate coalitions. Since each agent can gain at most one friend from strangers becoming friends, the best that any of these agents can do in separate coalitions is to be in a coalition with only their stranger and that they become friends with their stranger, giving them a utility of 6. Since the 3-clique trio of known friends provides 12 utility to all three members, any coalition which places all three members in separate coalitions is blocked by the 3-clique. Suppose instead that two members of a 3-clique are in one coalition and the third member is in another coalition. If the two clique members are in a pair coalition, then we've already established the 3-clique blocks the partition. What if the two clique members are in a coalition with their strangers instead? In a best-case scenario, such a partition gives the two clique members 11 utility, because they're in a coalition with two friends and at least one enemy. Since the trio provides 12 utility, it once again blocks any such partition. Thus we conclude that all possibly stable partitions must place all members of a 3-clique of known friends in the same coalition.

This reduces the number of partitions that may be possibly stable to two: the partition where each trio of known friends is in a 3-clique coalition (Figure 1 (b)), and the partition where the grand coalition forms (Figure 1 (c)). If all strangers become enemies, then the partition of two 3-cliques is SCS, and therefore CS, because all agents are in a coalition with all of their friends and none of their enemies, meaning that it is P-CS. Since no agent has any friends outside of their coalition and no enemies inside their coalition, every agent is in the best possible coalition under this resolution of strangers. The partition of two 3-cliques is not N-CS, because it is possibly blocked by the grand coalition when all strangers become friends. Now suppose that all strangers become friends and the grand coalition forms. Since all agents have three friends, no coalition with fewer than four agents can possibly block the grand coalition, since all agents in coalitions with less than 4 agents would lose at least one friend; however, since no two agents have the same stranger-turned-friend, no coalition of four or even five agents can break away without at least one agent losing a friend. Thus no coalition with four or five agents can block the grand coalition. Since there are only six agents in the partition, we conclude that the grand coalition is CS in the case where all strangers become friends, so it is P-CS. The grand coalition is not N-CS, because it is blocked by the partition of two 3-cliques when all strangers become enemies.

We've shown that all P-CS partitions must place all members of the known friend 3-cliques in the same coalition, reducing the number of partitions we must examine further to two. We went on to show that both of these partitions: the partition of two 3-cliques and the partition where the grand coalition forms are both P-CS, but neither is N-CS. Thus, we conclude that no N-CS partition exists for a FOHGS based upon the graph shown in Figure 1, thereby proving that N-CS partitions are not guaranteed to exist for FOGHS, even when relations are symmetric, agents have strictly more known friends than strangers, and at most one stranger.

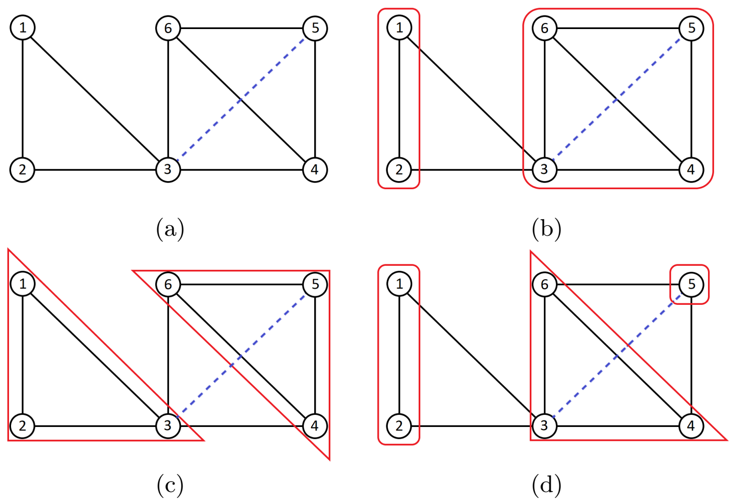

Fig. 2. EOHGS with no N-CS partition.

Proof for Theorem 10.

Proof. Consider a game represented by the graph in Figure 2 (a). Since we are dealing with EOHGS, any P-CS partition must only contain coalitions that are possibly cliques, since the presence of a single known enemy will guarantee that an agent's utility is negative, meaning that they benefit by leaving to become a singleton. Because of this, we focus only on partitions whose coalitions are all possible cliques. Since each friend in a coalition increases an agent's utility by 1, each agent's utility is maximized when they belong to the largest clique they are a member of. As a result, a P-CS partition will comprise a set of cliques which are possibly maximal for all their members. Based on this, we can reduce the partitions we consider to those described in Figures 2 (b), (c), and (d).

The partition described in Figure 2 (b) is P-CS. This is because agents 3—6 form a 4-clique when agents 3 and 5 are friends, which is the largest possible clique in the graph. The remaining agents 1 and 2 form a pair, since this is the largest clique they can form when 3 has joined the 4-clique. If 3 and 5 become enemies then {3} and {5} block {3,4,5,6}, so it is not N-CS.

Figure 2 (c) describes a pair of 3 cliques. If agents 3 and 5 are enemies, then the maximal clique size for all agents is 3, with the possible 3-cliques being

{1,2,3}, {3,4,6}, and {4,5,6}. Since the clique sizes are the same, agents 3, 4, and 6 are indifferent between the two cliques they each belong to. Agents 1, 2, and 5, meanwhile, only belong to a single 3-clique, so they prefer to be in the one 3-clique they belong to over any other possible coalition. As a result, the partition {{1,2,3}, {4,5,6}} is CS because no subset of the game's agents can form a large clique to block the partition. However, the partition described in Figure 2 (d) is also CS, because while agents 1, 2, and 5 can be better off in a 3-clique than as a pair and singleton respectively, the other agents of the 3-clique they belong to do not benefit from the change, so neither {1,2,3} nor {4,5,6} blocks the partition. Thus, whenever agents 3 and 5 are enemies, both of these partitions are CS, meaning that they are both P-CS. However, they are not N-CS, because both of these partitions are blocked by {3,4,5,6} if 3 and 5 become friends instead of enemies. Thus, none of the three P-CS partitions are N-CS, meaning that no N-CS partition exists for the EOHG instance described in Figure 2 (a).

If agents 3 and 5 become friends, then the partition shown in Figure 2 (b) is the only SCS partition, but if they become enemies, then either the partition in Figure 2 (c) or (d) can be CS, though (c) is SCS, while (d) is not.