A. For what values of the real number a does the quadratic equation

$$x^2 + ax + a = 1$$

have distinct real roots?

(a) $a \neq 2$; (b) $a > 2$; (c) $a = 2$; (d) all values of a.

B. The graph of $y = \sin x$ is reflected first in the line $x = \pi$ and then in the line $y = 2$. The resulting graph has equation

(a) $y = \cos x$; (b) $y = 2 + \sin x$; (c) $y = 4 + \sin x$; (d) $y = 2 - \cos x$.

Turn over

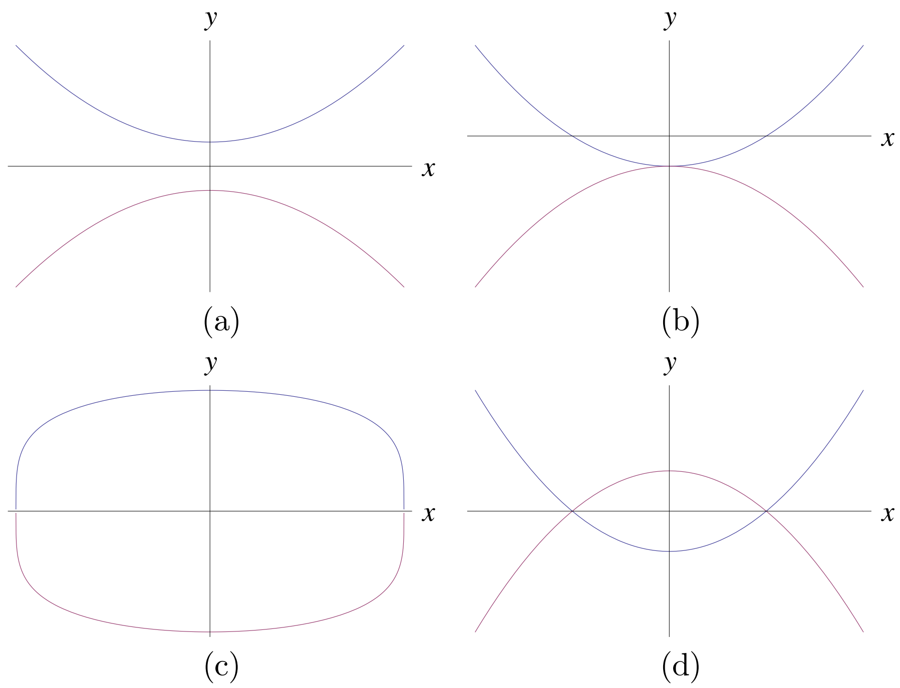
C. The functions f, g and h are related by

$$f'(x) = g(x+1), \qquad g'(x) = h(x-1).$$

It follows that $f''(2x)$ equals

(a) $h(2x+1)$; (b) $2h'(2x)$; (c) $h(2x)$; (d) $4h(2x)$.

D. Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?



E. The expression

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[(2x-1)^4(1-x)^5\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+1)^4\left(3x^2-2\right)^2\right]$$

is a polynomial of degree

(a) 9; (b) 8; (c) 7; (d) less than 7.

F. Three *positive* numbers a, b, c satisfy

$$\log_b a = 2, \qquad \log_b(c-3) = 3, \qquad \log_a(c+5) = 2.$$

This information

(a) specifies a uniquely.
(b) is satisfied by two values of a.
(c) is satisfied by infinitely many values of a.
(d) is contradictory.

Turn over

G. Let $n \geqslant 2$ be an integer and $p_n(x)$ be the polynomial

$$p_n(x) = (x-1) + (x-2) + \cdots + (x-n).$$

What is the remainder when $p_n(x)$ is divided by $p_{n-1}(x)$?

(a) $\frac{n}{2}$; (b) $\frac{n+1}{2}$; (c) $\frac{n^2+n}{2}$; (d) $\frac{-n}{2}$.

H. The area bounded by the graphs

$$y = \sqrt{2 - x^2} \qquad \text{and} \qquad x + \left(\sqrt{2} - 1\right) y = \sqrt{2}$$

equals

(a) $\frac{\sin\sqrt{2}}{\sqrt{2}}$; (b) $\frac{\pi}{4} - \frac{1}{\sqrt{2}}$; (c) $\frac{\pi}{2\sqrt{2}}$; (d) $\frac{\pi^2}{6}$.

I. The function $F(k)$ is defined for positive integers by $F(1) = 1$, $F(2) = 1$, $F(3) = -1$ and by the identities

$$F(2k) = F(k), \qquad F(2k+1) = F(k)$$

for $k \geqslant 2$. The sum

$$F(1) + F(2) + F(3) + \cdots + F(100)$$

equals

(a) -15; (b) 28; (c) 64; (d) 81.

J. For a real number x we denote by $[x]$ the largest integer less than or equal to x.

Let n be a natural number. The integral

$$\int_0^n [2^x] \, \mathrm{d}x$$

equals

(a) $\log_2((2^n-1)!)$; (b) $n2^n - \log_2((2^n)!)$; (c) $n2^n$; (d) $\log_2((2^n)!)$,

where $k! = 1 \times 2 \times 3 \times \cdots \times k$ for a positive integer k.

Turn over

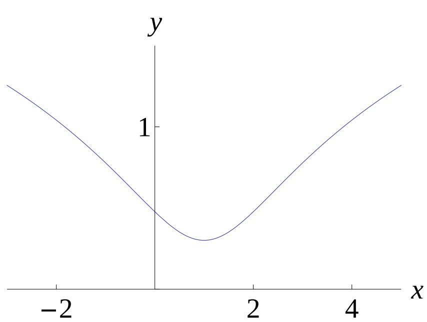
A. The inequality

$$x^4 < 8x^2 + 9$$

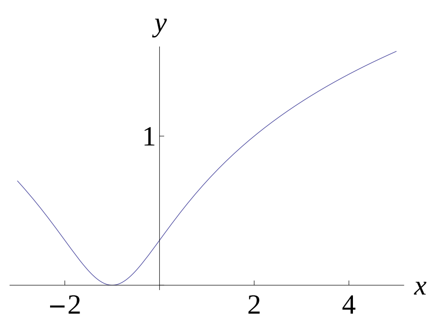
is satisfied precisely when

(a) $-3 < x < 3$; (b) $0 < x < 4$; (c) $1 < x < 3$; (d) $-1 < x < 9$; (e) $-3 < x < -1$.

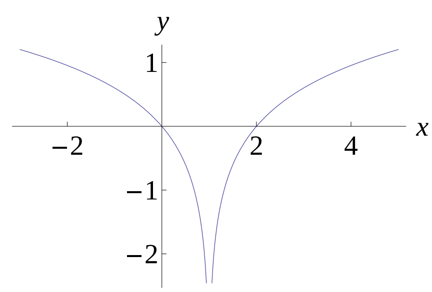
B. The graph of the function $y = \log_{10}(x^2 - 2x + 2)$ is sketched in



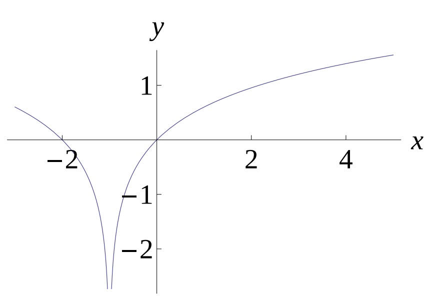
(a)



(b)



(c)



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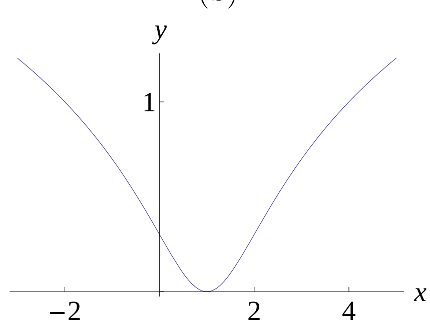


(e)

Turn over

C. The cubic

$$y = kx^3 - (k+1)x^2 + (2-k)x - k$$

has a turning point, that is a minimum, when $x = 1$ precisely for

(a) $k > 0$, (b) $0 < k < 1$, (c) $k > \frac{1}{2}$, (d) $k < 3$, (e) all values of k.

D. The reflection of the point $(1, 0)$ in the line $y = mx$ has coordinates

(a) $\left(\frac{m^2+1}{m^2-1}, \frac{m}{m^2-1}\right)$, (b) $(1, m)$, (c) $(1-m, m)$,

(d) $\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$, (e) $(1-m^2, m)$.

E. As x varies over the real numbers, the largest value taken by the function

$$\left(4\sin^2 x + 4\cos x + 1\right)^2$$

equals

(a) $17+12\sqrt{2}$, (b) 36, (c) $48\sqrt{2}$, (d) $64-12\sqrt{3}$, (e) 81.

F. The functions S and T are defined for real numbers by

$$S(x) = x + 1, \qquad \text{and} \qquad T(x) = -x.$$

The function S is applied s times and the function T is applied t times, *in some order*, to produce the function

$$F(x) = 8 - x.$$

It is possible to deduce that:

(a) $s = 8$ and $t = 1$.
(b) s is odd and t is even.
(c) s is even and t is odd.
(d) s and t are powers of 2.
(e) none of the above.

Turn over

G. Let n be a positive integer. The coefficient of x^3y^5 in the expansion of

$$(1 + xy + y^2)^n$$

equals

(a) n, (b) 2^n, (c) $\binom{n}{3}\binom{n}{5}$, (d) $4\binom{n}{4}$, (e) $\binom{n}{8}$.

H. The function $F(n)$ is defined for all positive integers as follows: $F(1) = 0$ and for all $n \geqslant 2$,

$$\begin{aligned}
F(n) &= F(n-1) + 2 && \text{if 2 divides } n \text{ but 3 does not divide } n; \\
F(n) &= F(n-1) + 3 && \text{if 3 divides } n \text{ but 2 does not divide } n; \\
F(n) &= F(n-1) + 4 && \text{if 2 and 3 both divide } n; \\
F(n) &= F(n-1) && \text{if neither 2 nor 3 divides } n.
\end{aligned}$$

The value of $F(6000)$ equals

(a) 9827, (b) 10121, (c) 11000, (d) 12300, (e) 12352.

I. The graph of the function
$$y = 2^{x^2-4x+3}$$
can be obtained from the graph of $y = 2^{x^2}$ by

(a) a stretch parallel to the y-axis followed by a translation parallel to the y-axis.
(b) a translation parallel to the x-axis followed by a stretch parallel to the y-axis.
(c) a translation parallel to the x-axis followed by a stretch parallel to the x-axis.
(d) a translation parallel to the x-axis followed by reflection in the y-axis.
(e) reflection in the y-axis followed by translation parallel to the y-axis.

J. For all real numbers x, the function $f(x)$ satisfies
$$6 + f(x) = 2f(-x) + 3x^2 \left(\int_{-1}^{1} f(t)\,\mathrm{d}t \right).$$
It follows that $\int_{-1}^{1} f(x)\,\mathrm{d}x$ equals

(a) 4, (b) 6, (c) 11, (d) $\dfrac{27}{2}$, (e) 23.

Turn over

A. Pick a whole number.

Add one.

Square the answer.

Multiply the answer by four.

Subtract three.

Which of the following statements are true regardless of which starting number is chosen?

I	The final answer is odd.
II	The final answer is one more than a multiple of three.
III	The final answer is one more than a multiple of eight.
IV	The final answer is not prime.
V	The final answer is not one less than a multiple of three.

(a) **I**, **II**, and **V**, (b) **I** and **IV**, (c) **II** and **V**,
(d) **I**, **III**, and **V**, (e) **I** and **V**.

B. Let

$$f(x) = (x + a)^n$$

where a is a real number and n is a positive whole number, and $n \geqslant 2$. If $y = f(x)$ and $y = f'(x)$ are plotted on the same axes, the number of intersections between $f(x)$ and $f'(x)$ will

(a) always be odd, (b) always be even, (c) depend on a but not n,
(d) depend on n but not a, (e) depend on both a and n.

Turn over

C. Which of the following are true for all real values of x? All arguments are in radians.

I	$\sin\left(\frac{\pi}{2}+x\right)=\cos\left(\frac{\pi}{2}-x\right)$
II	$2+2\sin(x)-\cos^2(x)\geqslant 0$
III	$\sin\left(x+\frac{3\pi}{2}\right)=\cos(\pi-x)$
IV	$\sin(x)\cos(x)\leqslant\frac{1}{4}$

(a) **I** and **II**, (b) **I** and **III**, (c) **II** and **III**,
(d) **III** and **IV**, (e) **II** and **IV**.

D. Let

$$f(x)=\int_0^1 (xt)^2 \mathrm{d}t, \qquad \text{and} \qquad g(x)=\int_0^x t^2 \mathrm{d}t.$$

Let $A>0$. Which of the following statements is true?
(a) $g(f(A))$ is always bigger than $f(g(A))$.
(b) $f(g(A))$ is always bigger than $g(f(A))$.
(c) They are always equal.
(d) $f(g(A))$ is bigger if $A<1$, and $g(f(A))$ is bigger if $A>1$.
(e) $g(f(A))$ is bigger if $A<1$ and $f(g(A))$ is bigger if $A>1$.

E. In the interval $0 \leqslant x \leqslant 2\pi$, the equation

$$\sin(2\cos(2x) + 2) = 0$$

has exactly

(a) 2 solutions, (b) 3 solutions, (c) 4 solutions, (d) 6 solutions, (e) 8 solutions.

F. For a real number x we denote by $\lfloor x \rfloor$ the largest integer less than or equal to x. Let

$$f(x) = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor.$$

The smallest number of equal width strips for which the trapezium rule produces an **overestimate** for the integral

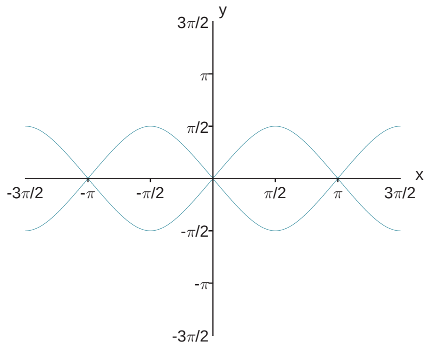
$$\int_0^5 f(x)\mathrm{d}x$$

is

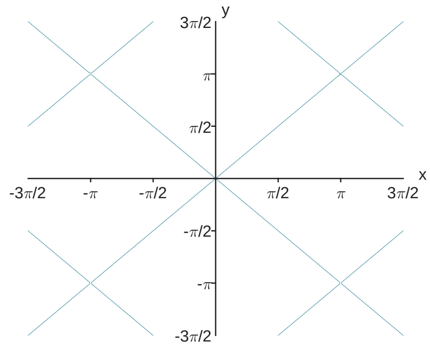
(a) 2, (b) 3, (c) 4, (d) 5, (e) it never produces an overestimate.

Turn over

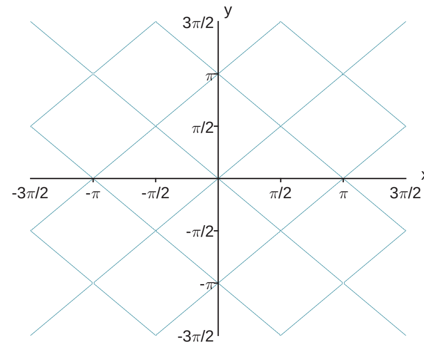
G. The graph of $\cos^2(x) = \cos^2(y)$ is sketched in



(a)



(b)



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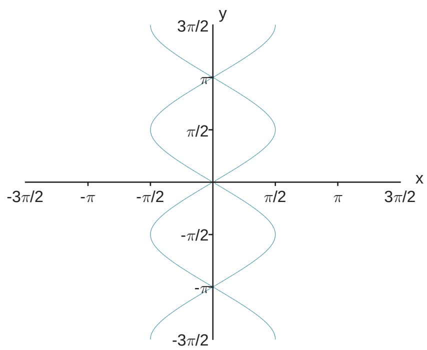
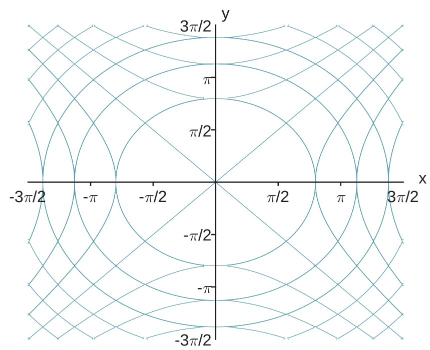


(d)



(e)

H. How many distinct solutions does the following equation have?

$$\log_{x^2+2}(4 - 5x^2 - 6x^3) = 2$$

(a) None, (b) 1, (c) 2, (d) 4, (e) Infinitely many.

I. Into how many regions is the plane divided when the following equations are graphed, not considering the axes?

$$y = x^3$$
$$y = x^4$$
$$y = x^5$$

(a) 6, (b) 7, (c) 8, (d) 9, (e) 10.

J. Which is the largest of the following numbers?

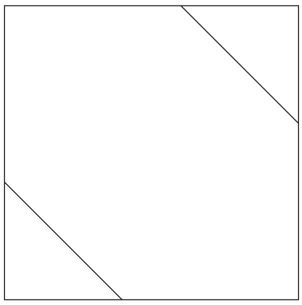
(a) $\dfrac{\sqrt{7}}{2}$, (b) $\dfrac{5}{4}$, (c) $\dfrac{\sqrt{10!}}{3(6!)}$, (d) $\dfrac{\log_2(30)}{\log_3(85)}$, (e) $\dfrac{1+\sqrt{6}}{3}$.

Turn over

A. A sequence (a_n) has first term $a_1 = 1$, and subsequent terms defined by $a_{n+1} = la_n$ for $n \geqslant 1$. What is the product of the first 15 terms of the sequence?

(a) l^{14}, (b) $15 + l^{14}$, (c) $\dfrac{1 - l^{15}}{1 - l}$, (d) l^{105}, (e) $15 + l^{105}$.

B. An irregular hexagon with all sides of equal length is placed inside a square of side length 1, as shown below (not to scale). What is the length of one of the hexagon sides?



(a) $\sqrt{2} - 1$, (b) $2 - \sqrt{2}$, (c) 1, (d) $\dfrac{\sqrt{2}}{2}$, (e) $2 + \sqrt{2}$.

Turn over

C. The origin lies inside the circle with equation

$$x^2 + ax + y^2 + by = c$$

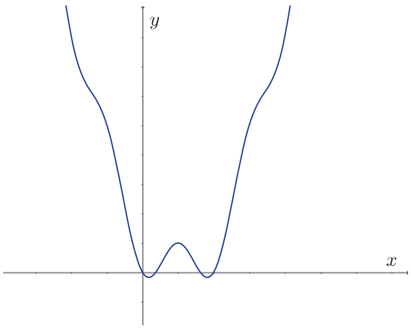
precisely when

(a) $c > 0$, (b) $a^2+b^2 > c$, (c) $a^2+b^2 < c$, (d) $a^2+b^2 > 4c$, (e) $a^2+b^2 < 4c$.

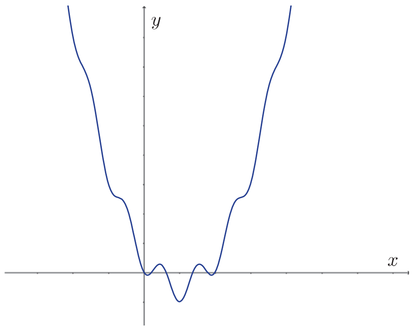
D. How many solutions does $\cos^n(x) + \cos^{2n}(x) = 0$ have in the range $0 \leqslant x \leqslant 2\pi$ for an integer $n \geqslant 1$?

(a) 1 for all n, (b) 2 for all n, (c) 3 for all n,
(d) 2 for even n and 3 for odd n, (e) 3 for even n and 2 for odd n.

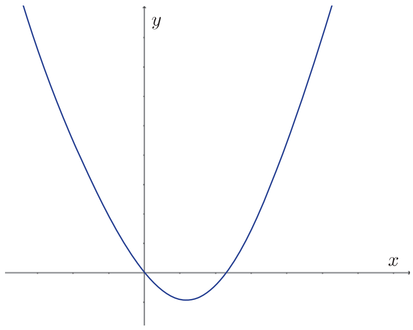
E. The graph of $y = (x-1)^2 - \cos(\pi x)$ is drawn in



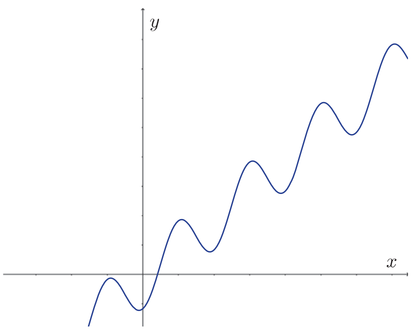
(a)



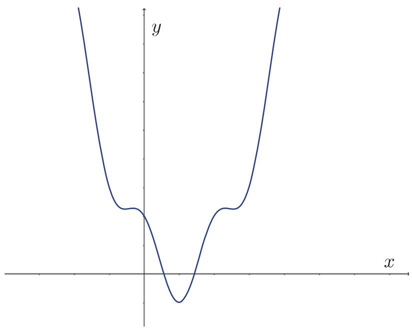
(b)



(c)



(d)



(e)

F. Let n be a positive integer. Then x^2+1 is a factor of

$$(3+x^4)^n - (x^2+3)^n(x^2-1)^n$$

for

(a) all n, (b) even n, (c) odd n, (d) $n \geqslant 3$, (e) no values of n.

Turn over

G. The sequence (x_n), where $n \geqslant 0$, is defined by $x_0 = 1$ and

$$x_n = \sum_{k=0}^{n-1} x_k \qquad \text{for } n \geqslant 1.$$

The sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

equals

(a) 1, (b) $\frac{6}{5}$, (c) $\frac{8}{5}$, (d) 3, (e) $\frac{27}{5}$.

H. Consider two functions

$$f(x) = a - x^2$$
$$g(x) = x^4 - a.$$

For precisely which values of $a > 0$ is the area of the region bounded by the x-axis and the curve $y = f(x)$ bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$?

(a) all values of a, (b) $a > 1$, (c) $a > \frac{6}{5}$,

(d) $a > \left(\frac{4}{3}\right)^{\frac{3}{2}}$, (e) $a > \left(\frac{6}{5}\right)^{4}$.

I. Let a and b be positive real numbers. If $x^2 + y^2 \leqslant 1$ then the largest that $ax + by$ can equal is

(a) $\dfrac{1}{a}+\dfrac{1}{b}$, (b) $\max(a, b)$, (c) $\sqrt{a^2 + b^2}$, (d) $a+b$, (e) a^2+ab+b^2.

J. Let $n > 1$ be an integer. Let $\Pi(n)$ denote the number of distinct prime factors of n and let $x(n)$ denote the final digit of n. For example, $\Pi(8) = 1$ and $\Pi(6) = 2$. Which of the following statements is false?

(a) If $\Pi(n) = 1,$ there are some values of $x(n)$ that mean n cannot be prime,

(b) If $\Pi(n) = 1,$ there are some values of $x(n)$ that mean n must be prime,

(c) If $\Pi(n) = 1,$ there are values of $x(n)$ which are impossible,

(d) If $\Pi(n) + x(n) = 2,$ we cannot tell if n is prime,

(e) If $\Pi(n) = 2,$ all values of $x(n)$ are possible.

Turn over

A. Let

$$f(x) = 2x^3 - kx^2 + 2x - k.$$

For what values of the real number k does the graph $y = f(x)$ have two distinct real stationary points?

(a) $-2\sqrt{3} < k < 2\sqrt{3}$
(b) $k < -2\sqrt{3}$ or $2\sqrt{3} < k$
(c) $k < -\sqrt{21} - 3$ or $\sqrt{21} - 3 < k$
(d) $-\sqrt{21} - 3 < k < \sqrt{21} - 3$
(e) all values of k.

B. The minimum value achieved by the function

$$f(x) = 9\cos^4 x - 12\cos^2 x + 7$$

equals

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7.

Turn over

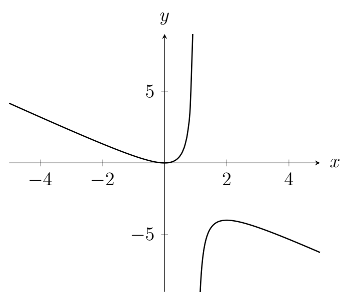
C. A sequence (a_n) has the property that

$$a_{n+1} = \frac{a_n}{a_{n-1}}$$

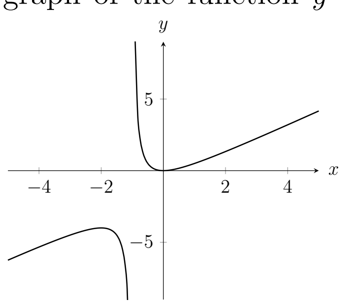
for every $n \geqslant 2$. Given that $a_1 = 2$ and $a_2 = 6$, what is a_{2017}?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 2 (e) 3.

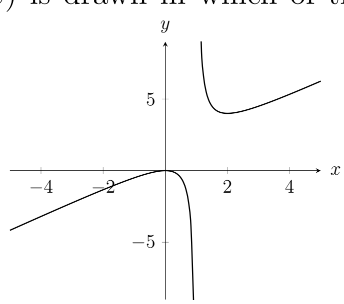
D. The diagram below shows the graph of $y = f(x)$.



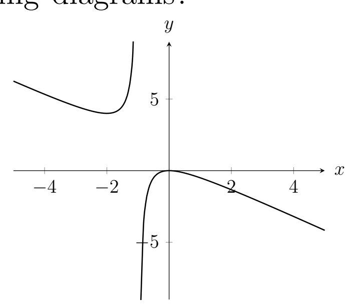
The graph of the function $y = -f(-x)$ is drawn in which of the following diagrams?



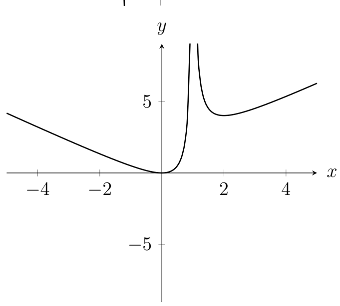
(a)



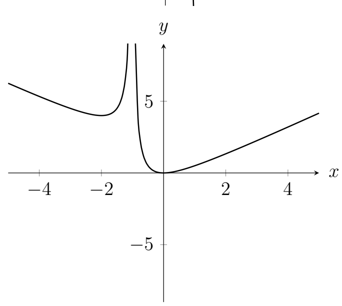
(b)



(c)



(d)



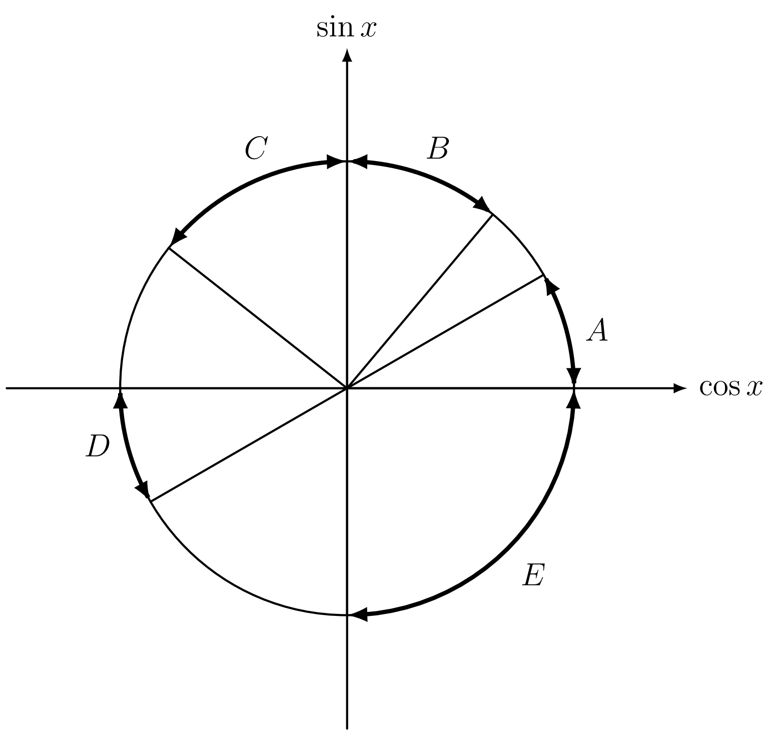
(e)

E. Let a and b be *positive integers* such that $a + b = 20$. What is the maximum value that $a^2 b$ can take?

(a) 1000 (b) 1152 (c) 1176 (d) 1183 (e) 1196.

F. The picture below shows the unit circle, where each point has coordinates $(\cos x, \sin x)$ for some x. Which of the marked arcs corresponds to

$$\tan x < \cos x < \sin x \ ?$$



(a) A (b) B (c) C (d) D (e) E.

Turn over

G. For all θ in the range $0 \leqslant \theta < 2\pi$ the line

$$(y-1)\cos\theta = (x+1)\sin\theta$$

divides the disc $x^2 + y^2 \leqslant 4$ into two regions. Let $A(\theta)$ denote the area of the larger region.

Then $A(\theta)$ achieves its maximum value at

(a) one value of θ (b) two values of θ (c) three values of θ

(d) four values of θ (e) all values of θ.

H. In this question a and b are real numbers, and a is non-zero.
When the polynomial $x^2 - 2ax + a^4$ is divided by $x + b$ the remainder is 1.
The polynomial $bx^2 + x + 1$ has $ax - 1$ as a factor.

It follows that b equals

(a) 1 only (b) 0 or -2 (c) 1 or 2 (d) 1 or 3 (e) -1 or 2.

I. Let $a, b, c > 0$ and $a \neq 1$. The equation

$$\log_b\left((b^x)^x\right) + \log_a\left(\frac{c^x}{b^x}\right) + \log_a\left(\frac{1}{b}\right)\log_a(c) = 0$$

has a repeated root when

(a) $b^2 = 4ac$ (b) $b = \dfrac{1}{a}$ (c) $c = \dfrac{b}{a}$ (d) $c = \dfrac{1}{b}$ (e) $a = b = c$.

J. Which of these integrals has the largest value? You are not expected to calculate the exact value of any of these.

(a) $\displaystyle\int_0^2 (x^2-4)\sin^8(\pi x)\,\mathrm{d}x$ (b) $\displaystyle\int_0^{2\pi} (2+\cos x)^3\,\mathrm{d}x$ (c) $\displaystyle\int_0^{\pi} \sin^{100} x\,\mathrm{d}x$

(d) $\displaystyle\int_0^{\pi} (3-\sin x)^6\,\mathrm{d}x$ (e) $\displaystyle\int_0^{8\pi} 108(\sin^3 x - 1)\,\mathrm{d}x$.

Turn over

A. The area of the region bounded by the curve $y = \sqrt{x}$, the line $y = x - 2$ and the x-axis equals

(a) 2, (b) $\dfrac{5}{2}$, (c) 3, (d) $\dfrac{10}{3}$, (e) $\dfrac{16}{3}$.

B. The function $y = e^{kx}$ satisfies the equation

$$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - y\right) = y\frac{\mathrm{d}y}{\mathrm{d}x}$$

for
(a) no values of k,
(b) exactly one value of k,
(c) exactly two distinct values of k,
(d) exactly three distinct values of k,
(e) infinitely many distinct values of k.

Turn over

C. Let a, b, c and d be real numbers. The two curves $y = ax^2 + c$ and $y = bx^2 + d$ have exactly two points of intersection precisely when

(a) $\dfrac{a}{b} < 1$, (b) $\dfrac{a}{b} < \dfrac{c}{d}$, (c) $a < b$, (d) $c < d$, (e) $(d-c)(a-b) > 0$.

D. If $f(x) = x^2 - 5x + 7$, what are the coordinates of the minimum of $y = f(x-2)$?

(a) $\left(\dfrac{5}{2}, \dfrac{3}{4}\right)$, (b) $\left(\dfrac{9}{2}, \dfrac{3}{4}\right)$, (c) $\left(\dfrac{1}{2}, \dfrac{3}{4}\right)$, (d) $\left(\dfrac{9}{2}, \dfrac{-5}{4}\right)$, (e) $\left(\dfrac{5}{2}, \dfrac{-5}{4}\right)$.

E. A circle of radius 2, centred on the origin, is drawn on a grid of points with integer coordinates. Let n be the number of grid points that lie within or on the circle. What is the smallest amount the radius needs to increase by for there to be $2n-5$ grid points within or on the circle?

(a) $\sqrt{5}-2$, (b) $\sqrt{6}-2$, (c) $\sqrt{8}-2$, (d) 1, (e) $\sqrt{8}$.

F. A particle moves in the xy-plane, starting at the origin $(0,0)$. At each turn, the particle may move in one of two ways:

- it may move two to the right and one up, that is, it may be translated by the vector $(2,1)$, or
- it may move one to the right and two up, that is, it may be translated by the vector $(1,2)$.

What is the closest the particle may come to the point $(25,75)$?

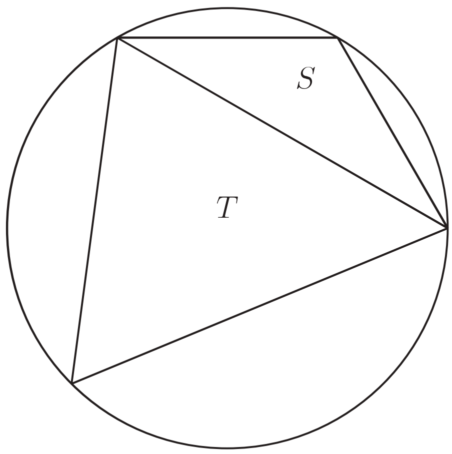
(a) 0, (b) $5\sqrt{5}$, (c) $2\sqrt{53}$, (d) 25, (e) 35.

Turn over

G. The parabolas with equations $y = x^2 + c$ and $y^2 = x$ touch (that is, meet tangentially) at a single point. It follows that c equals

(a) $\dfrac{1}{2\sqrt{3}}$, (b) $\dfrac{3}{4\sqrt[3]{4}}$, (c) $\dfrac{-1}{2}$, (d) $\sqrt{5}-\sqrt{3}$, (e) $\sqrt{\dfrac{2}{3}}$.

H. Two triangles S and T are inscribed in a circle, as shown in the diagram below.



The triangles have respective areas s and t and S is the smaller triangle so that $s < t$. The smallest value that

$$\frac{4s^2 + t^2}{5st}$$

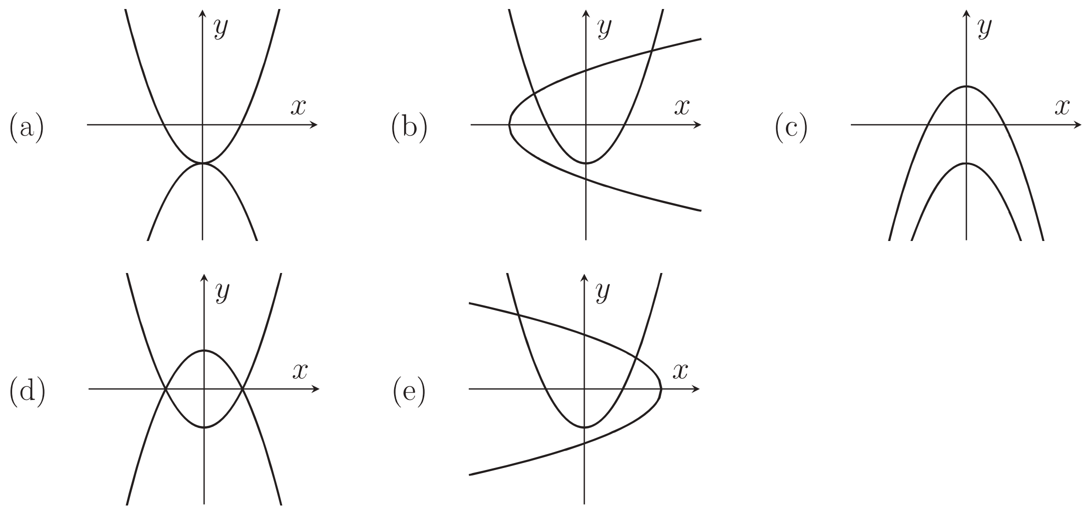
can equal is

(a) $\dfrac{2}{5}$, (b) $\dfrac{3}{5}$, (c) $\dfrac{4}{5}$, (d) 1, (e) $\dfrac{3}{2}$.

I. A sketch of the curve

$$(x^8 + 4yx^6 + 6y^2x^4 + 4y^3x^2 + y^4)^2 = 1$$

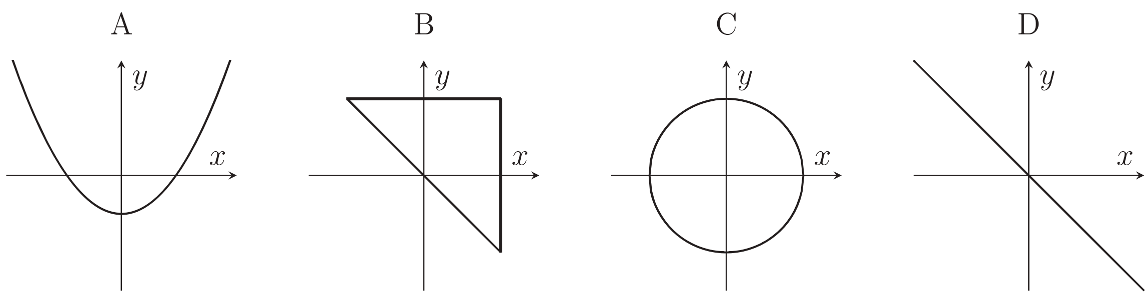
is given below in



J. Which of the following could be the sketch of a curve

$$p(x) + p(y) = 0$$

for some polynomial p?



(a) A and D, but not B or C;
(b) A and B, but not C or D;
(c) C and D, but not A or B;
(d) A, C and D, but not B;
(e) A, B and C, but not D.

Turn over

A. The equation

$$x^3 - 300x = 3000$$

has

(a) no real solutions.
(b) exactly one real solution.
(c) exactly two real solutions.
(d) exactly three real solutions.
(e) infinitely many real solutions.

B. The product of a square number and a cube number is

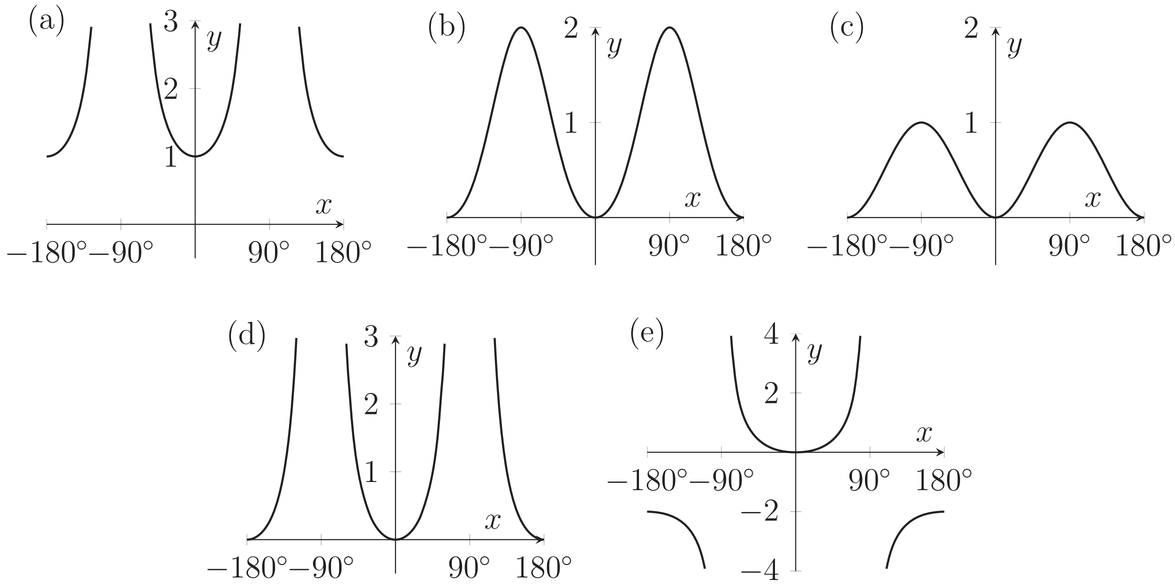
(a) always a square number, and never a cube number.
(b) always a cube number, and never a square number.
(c) sometimes a square number, and sometimes a cube number.
(d) never a square number, and never a cube number.
(e) always a cube number, and always a square number.

Turn over

C. The graph of

$$y = \sin^2 x + \sin^4 x + \sin^6 x + \sin^8 x + \cdots$$

is sketched in



D. The area between the parabolas with equations $y = x^2 + 2ax + a$ and $y = a - x^2$ equals 9. The possible values of a are

(a) $a = 1$, (b) $a = -3$ or $a = 3$, (c) $a = -3$,

(d) $a = -1$ or $a = 1$, (e) $a = 1$ or $a = 3$.

E. The graph of

$$\sin y - \sin x = \cos^2 x - \cos^2 y$$

(a) is empty.
(b) is non-empty but includes no straight lines.
(c) includes precisely one straight line.
(d) includes precisely two straight lines.
(e) includes infinitely many straight lines.

F. In the interval $0 \leqslant x < 360°$, the equation

$$\sin^3 x + \cos^2 x = 0$$

has

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4

solutions.

Turn over

G. Let $a, b, c > 0$. The equations

$$\log_a b = c, \qquad \log_b a = c + \frac{3}{2}, \qquad \log_c a = b,$$

(a) specify a, b and c uniquely.
(b) specify c uniquely but have infinitely many solutions for a and b.
(c) specify c and a uniquely but have infinitely many solutions for b.
(d) specify a and b uniquely but have infinitely many solutions for c.
(e) have no solutions for a, b and c.

H. The triangle ABC is right-angled at B and the side lengths are positive numbers in geometric progression. It follows that $\tan \angle BAC$ is either

(a) $\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{1-\sqrt{5}}{2}}$, (b) $\sqrt{\frac{1+\sqrt{3}}{2}}$ or $\sqrt{\frac{\sqrt{3}-1}{2}}$, (c) $\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{\sqrt{5}-1}{2}}$

(d) $-\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{1+\sqrt{5}}{2}}$, (e) $\sqrt{\frac{1+\sqrt{3}}{2}}$ or $\sqrt{\frac{1-\sqrt{3}}{2}}$.

I. The positive real numbers x and y satisfy $0 < x < y$ and

$$x2^x = y2^y.$$

for
(a) no pairs x and y.
(b) exactly one pair x and y.
(c) exactly two pairs x and y.
(d) exactly four pairs x and y.
(e) infinitely many pairs x and y.

J. An equilateral triangle has centre O and side length 1. A straight line through O intersects the triangle at two distinct points P and Q. The minimum possible length of PQ is

$$\text{(a)}\ \frac{1}{3},\quad \text{(b)}\ \frac{1}{2},\quad \text{(c)}\ \frac{\sqrt{3}}{3},\quad \text{(d)}\ \frac{2}{3},\quad \text{(e)}\ \frac{\sqrt{3}}{2}.$$

Turn over

A. A square has centre $(3,4)$ and one corner at $(1,5)$. Another corner is at

(a) $(1,3)$, (b) $(5,5)$, (c) $(4,2)$, (d) $(2,2)$, (e) $(5,2)$.

B. What is the value of $\displaystyle\int_0^1 (e^x - x)(e^x + x)\,\mathrm{d}x$?

(a) $\dfrac{3e^2-2}{6}$, (b) $\dfrac{3e^2+2}{6}$, (c) $\dfrac{2e^2-3}{6}$, (d) $\dfrac{3e^2-5}{6}$, (e) $\dfrac{e^2+3}{6}$.

Turn over

C. The sum

$$1 - 4 + 9 - 16 + \cdots + 99^2 - 100^2$$

equals

(a) -101 (b) -1000 (c) -1111 (d) -4545 (e) -5050.

D. The largest value achieved by $3\cos^2 x + 2\sin x + 1$ equals

(a) $\dfrac{11}{5}$, (b) $\dfrac{13}{3}$, (c) $\dfrac{12}{5}$, (d) $\dfrac{14}{9}$, (e) $\dfrac{12}{7}$.

E. A line is tangent to the parabola $y = x^2$ at the point (a, a^2) where $a > 0$. The area of the region bounded by the parabola, the tangent line, and the x-axis equals

(a) $\dfrac{a^2}{3}$, (b) $\dfrac{2a^2}{3}$, (c) $\dfrac{a^3}{12}$, (d) $\dfrac{5a^3}{6}$, (e) $\dfrac{a^4}{10}$.

F. Which of the following expressions is equal to $\log_{10}(10 \times 9 \times 8 \times \cdots \times 2 \times 1)$?

(a) $1 + 5\log_{10} 2 + 4\log_{10} 6$,
(b) $1 + 4\log_{10} 2 + 2\log_{10} 6 + \log_{10} 7$,
(c) $2 + 2\log_{10} 2 + 4\log_{10} 6 + \log_{10} 7$,
(d) $2 + 6\log_{10} 2 + 4\log_{10} 6 + \log_{10} 7$,
(e) $2 + 6\log_{10} 2 + 4\log_{10} 6$.

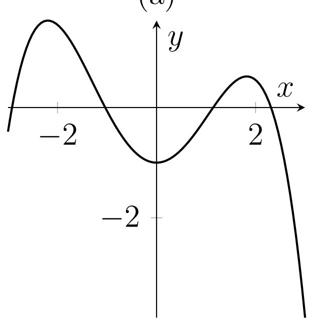
Turn over

G. A cubic has equation $y = x^3 + ax^2 + bx + c$ and has turning points at $(1, 2)$ and $(3, d)$ for some d. What is the value of d?

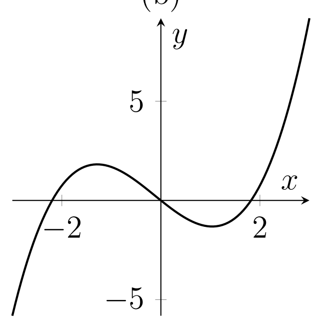
(a) -4, (b) -2, (c) 0, (d) 2, (e) 4.

H. The following five graphs are, in some order, plots of $y = f(x)$, $y = g(x)$, $y = h(x)$, $y = \dfrac{\mathrm{d}f}{\mathrm{d}x}$ and $y = \dfrac{\mathrm{d}g}{\mathrm{d}x}$; that is, three unknown functions and the derivatives of the first two of those functions. Which graph is a plot of $h(x)$?

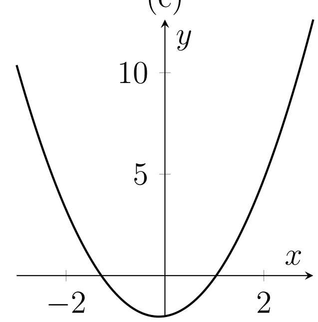
(a)



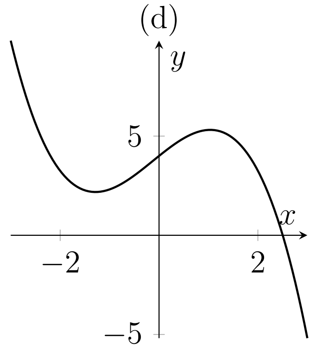
(b)



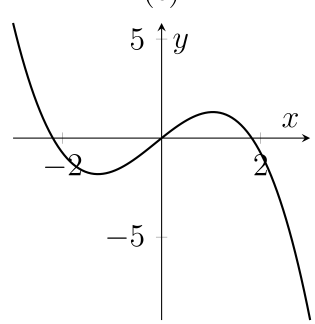
(c)



(d)



(e)



I. In the range $-90° < x < 90°$, how many values of x are there for which the sum to infinity

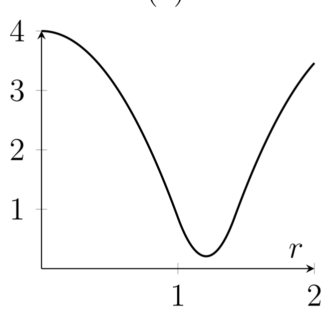
$$\frac{1}{\tan x} + \frac{1}{\tan^2 x} + \frac{1}{\tan^3 x} + \dots$$

equals $\tan x$?

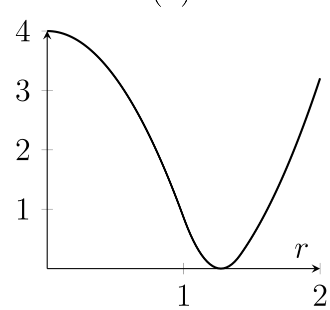
(a) 0 (b) 1 (c) 2 (d) 3 (e) 4.

J. Consider a square with side length 2 and centre $(0,0)$, and a circle with radius r and centre $(0,0)$. Let $A(r)$ be the area of the region that is inside the circle but outside the square, and let $B(r)$ be the area of the region that is inside the square but outside the circle. Which of the following is a sketch of $A(r) + B(r)$?

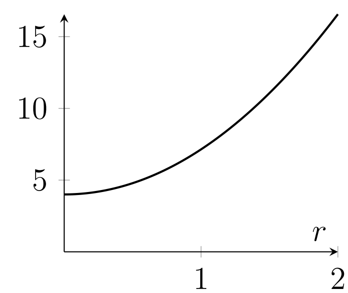
(a)



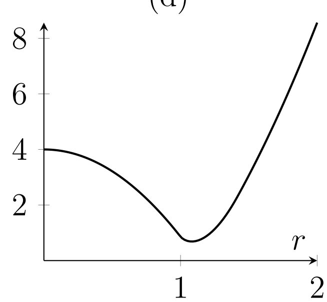
(b)



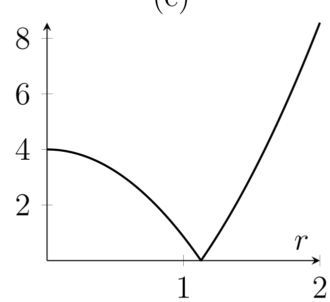
(c)



(d)



(e)



Turn over

A. A regular dodecagon is a 12-sided polygon with all sides the same length and all internal angles equal. If I construct a regular dodecagon by connecting 12 equally-spaced points on a circle of radius 1, then the area of this polygon is

(a) $6 + 3\sqrt{3}$, (b) $2\sqrt{2}$, (c) $3\sqrt{2}$, (d) $3\sqrt{3}$, (e) 3.

B. The positive number a satisfies

$$\int_0^a \left(\sqrt{x} + x^2\right) \, dx = 5$$

if

(a) $a = \left(\sqrt{21} - 1\right)^{1/3}$, (b) $a = \sqrt{3}$, (c) $a = 3^{2/3}$,
(d) $a = (\sqrt{6} - 1)^{2/3}$, (e) $a = 5^{2/3}$.

Turn over

C. Tangents to $y = e^x$ are drawn at (p, e^p) and (q, e^q). These tangents cross the x-axis at a and b respectively. It follows that, for all p and q,

(a) $pa = qb$,

(b) $p - a < q - b$,

(c) $p - a = q - b$,

(d) $p - a > q - b$,

(e) $p + q = a + b$.

D. The area of the region bounded by the curve $y = e^x$, the curve $y = 1 - e^x$, and the y-axis equals

(a) 0, (b) $1 - \ln 2$, (c) $\frac{1}{2} - \frac{1}{2}\ln 2$,
(d) $\ln 2 - 1$, (e) $1 - \ln \frac{1}{2}$.

[Note that $\ln x$ is alternative notation for $\log_e x$.]

E. Six vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$, $\mathbf{v}_6$ are each chosen to be either $\begin{pmatrix}1\\1\end{pmatrix}$ or $\begin{pmatrix}3\\2\end{pmatrix}$ with equal probability, with each choice made independently. The probability that the sum $\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4+\mathbf{v}_5+\mathbf{v}_6$ is equal to the vector $\begin{pmatrix}10\\8\end{pmatrix}$ is

(a) 0, (b) $\frac{3}{64}$, (c) $\frac{15}{64}$, (d) $\frac{1}{6}$, (e) $\frac{5}{16}$.

F. The tangent to the curve $y = x^3 - 3x$ at the point $(a, a^3 - 3a)$ also passes through the point $(2, 0)$ for precisely

(a) no values of a,

(b) one value of a,

(c) two values of a,

(d) three values of a,

(e) all values of a.

Turn over

G. The sum

$$\sin^2(1^\circ) + \sin^2(2^\circ) + \sin^2(3^\circ) + \cdots + \sin^2(89^\circ) + \sin^2(90^\circ)$$

is equal to

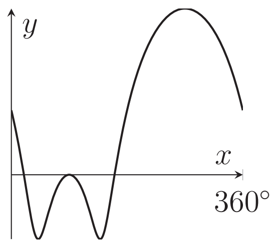
(a) 44, (b) $44\frac{1}{2}$, (c) 45, (d) $45\frac{1}{2}$, (e) 46.

H. Which of the following graphs shows

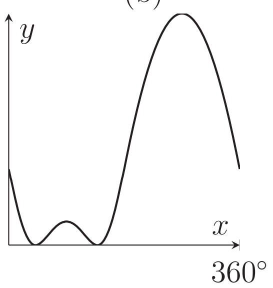
$$y = \log_2\left(9 - 8\sin x - 6\cos^2 x\right)$$

in the range $0 \leqslant x \leqslant 360^\circ$?

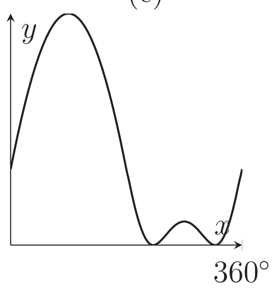
(a)



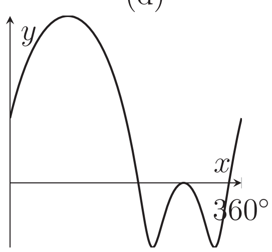
(b)



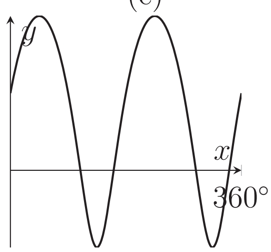
(c)



(d)



(e)



I. A sequence is defined by $a_0 = 2$ and then for $n \geqslant 1$, a_n is one more than the product of all previous terms (so $a_1 = 3$ and $a_2 = 7$, for example). It follows that for all $n \geqslant 1$,

(a) $a_n = 4a_{n-1} - 5$,

(b) $a_n = a_{n-1}\,(a_{n-1} - 1) + 1$,

(c) $a_n = 2a_{n-1}(a_{n-1} - 3) + 7$,

(d) $a_n = \dfrac{3}{2}n^2 - \dfrac{1}{2}n + 2$,

(e) None of the above.

J. Four distinct real numbers a, b, c, and d are used to define four points

$$A = (a, b), \quad B = (b, c), \quad C = (c, d), \quad D = (d, a).$$

The quadrilateral $ABCD$ has all four sides the same length

(a) if and only if $(a - b)^2 = (c - d)^2$,

(b) if and only if $(a - c)^2 = (b - d)^2$,

(c) if and only if $(a - d)^2 = (b - c)^2$,

(d) if and only if $a - b + c - d = 0$,

(e) for no values of a, b, c, d.

Turn over

A. How many real solutions x are there to the equation $x|x| + 1 = 3|x|$?

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.

[Note that $|x|$ is equal to x if $x \geqslant 0$, and equal to $-x$ otherwise.]

B. One hundred circles all share the same centre, and they are named C_1, C_2, C_3, and so on up to C_{100}. For each whole number n between 1 and 99 inclusive, a tangent to circle C_n crosses circle C_{n+1} at two points that are separated by a distance of 2. Given that C_1 has radius 1, it follows that the radius of C_{100} is

(a) 1, (b) 2, (c) $\sqrt{10}$, (d) 10, (e) 100.

Turn over

C. The equation $x^2 - 4kx + y^2 - 4y + 8 = k^3 - k$ is the equation of a circle

(a) for all real values of k.

(b) if and only if either $-4 < k < -1$ or $k > 1$.

(c) if and only if $k > 1$.

(d) if and only if $k < -1$.

(e) if and only if either $-1 < k < 0$ or $k > 1$.

D. A sequence has $a_0 = 3$, and then for $n \geqslant 1$ the sequence satisfies $a_n = 8\left(a_{n-1}\right)^4$. The value of a_{10} is

(a) $\dfrac{2^{(2^{20})}}{3}$, (b) $\dfrac{6^{(2^{20})}}{3}$, (c) $\dfrac{3^{(2^{20})}}{2}$, (d) $\dfrac{18^{(2^{20})}}{2}$, (e) $\dfrac{6^{(2^{20})}}{2}$.

E. If the expression $\left(x+1+\frac{1}{x}\right)^4$ is fully expanded term-by-term and like terms are collected together, there is one term which is independent of x. The value of this term is

(a) 10, (b) 14, (c) 19, (d) 51, (e) 81.

F. Given that

$$\sin(5\theta) = 5\sin\theta - 20(\sin\theta)^3 + 16(\sin\theta)^5$$

for all real θ, it follows that the value of $\sin(72°)$ is

(a) $\sqrt{\dfrac{5+\sqrt{5}}{8}}$, (b) 0, (c) $-\sqrt{\dfrac{5+\sqrt{5}}{8}}$,

(d) $\sqrt{\dfrac{5-\sqrt{5}}{8}}$, (e) $-\sqrt{\dfrac{5-\sqrt{5}}{8}}$.

Turn over

G. For all real n, it is the case that $n^4 + 1 = (n^2 + \sqrt{2}n + 1)(n^2 - \sqrt{2}n + 1)$. From this we may deduce that $n^4 + 4$ is

(a) never a prime number for any positive whole number n.

(b) a prime number for exactly one positive whole number n.

(c) a prime number for exactly two positive whole numbers n.

(d) a prime number for exactly three positive whole numbers n.

(e) a prime number for exactly four positive whole numbers n.

H. How many real solutions x are there to the following equation?

$$\log_2(2x^3 + 7x^2 + 2x + 3) = 3\log_2(x + 1) + 1$$

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.

I. Alice and Bob each toss five fair coins (each coin lands on either heads or tails, with equal probability and with each outcome independent of each other). Alice wins if strictly more of her coins land on heads than Bob's coins do, and we call the probability of this event p_1. The game is a draw if the same number of coins land on heads for each of Alice and Bob, and we call the probability of this event p_2. Which of the following is correct?

(a) $p_1 = \dfrac{193}{512}$ and $p_2 = \dfrac{63}{256}$.

(b) $p_1 = \dfrac{201}{512}$ and $p_2 = \dfrac{55}{256}$.

(c) $p_1 = \dfrac{243}{512}$ and $p_2 = \dfrac{13}{256}$.

(d) $p_1 = \dfrac{247}{512}$ and $p_2 = \dfrac{9}{256}$.

(e) $p_1 = \dfrac{1}{3}$ and $p_2 = \dfrac{1}{3}$.

J. The real numbers m and c are such that the equation

$$x^2 + (mx + c)^2 = 1$$

has a repeated root x, and also the equation

$$(x - 3)^2 + (mx + c - 1)^2 = 1$$

has a repeated root x (which is not necessarily the same value of x as the root of the first equation). How many possibilities are there for the line $y = mx + c$?

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.

Turn over